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DESIGN PROCEDURES FOR DOMINANT TYPE SYSTEMS WITH LARGE PARAMETER VARIATIONS

By Don Eugene Olson

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Department of Electrical Engineering
University of Colorado
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Abstract--This report presents design procedures for fourth-order dominant type systems with large plant parameter variations. The s-domain specifications of the system are assumed to be in the form of an acceptable dominant closed loop pole region and bounds on the location of the "far-off" closed loop poles. The design philosophy is to place compensation zeros within the acceptable dominant closed loop pole region such that the dominant closed loop poles remain within their prescribed region despite the large variations in the plant parameters. Design procedures are presented for variation in the plant gain factory only and for simultaneous variation in the plant gain factor and the plant poles. Finally, an approximate procedure is presented which considers simultaneous variation in the plant gain factor, the plant poles and a plant zero located on the real axis in the s-plane. In all cases, the design procedures are such as to minimize the sensitivity of the system to internal noise.

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CHAPTER I

PROBLEM STATEMENT AND DESIGN PHILOSOPHY

1.1 Statement of the Problem

The problem considered in this document may be stated as follows: 1). A single input-single output plant* P(s) has parameters (gain factor, poles and zeroes) which may lie or "slowly" vary within a given region in the s-plane. 2). The acceptable region of closed loop poles is specified in the s-plane. This acceptable region is determined by transforming the typical type domain specifications such as rise time, over-shoot and settling time into specifications on the location of the dominant poles of the system. This transformation of these specifications from the time domain to the pole-zero domain is considered by Barber.[1] 3). Linear time invariant compensation is to be chosen to satisfy the above specifications such that the effect of internal noise at the plant input

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*The term "plant" is used here in the commonly accepted sense in the control literature to denote the constrained part of the system whose output is the system output.

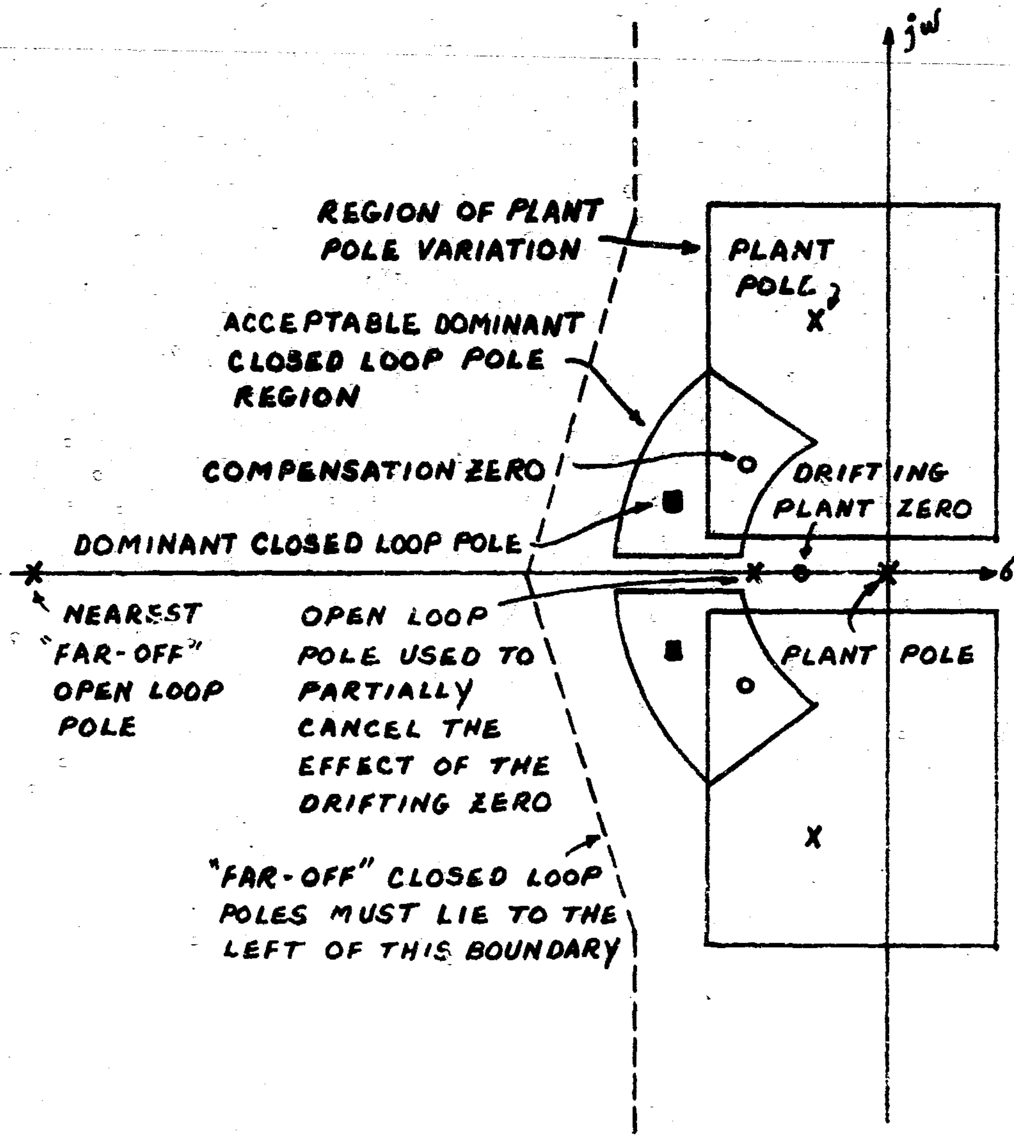
is minimized. The pictorial representation of a typical problem encountered in flight control is shown in Fig. 1.1.

1.2 State of the Art

This problem has been treated in the literature.[2,3] However, the design techniques presented have several deficiencies. The mapping of the region of plant parameter variation into the acceptable closed loop pole region is approximate and is only valid if the acceptable closed loop pole region is relatively small and well removed from the plant parameter variation in the s-plane. An additional deficiency is that the unavoidable large loop transmission bandwidth involved in using these design techniques results in a very unfavorable high frequency response to internal noise at the plant input.

A design procedure which greatly alleviates these deficiencies is the topic of a recent paper by Horowitz.[4] The design procedure presented in the present document is essentially that developed in the paper by Horowitz, expanded to take into account the effect of one of the "far-off" poles. Thus Horowitz uses a third order representation in deriving the dominant poles of the system whereas this treatment uses a fourth order representation. Chapter IV also

FIG. 1.1 STATEMENT OF THE PROBLEM



considers the additional effect of a drifting zero on the real axis.

1.3 Design Philosophy

The design procedure presented in this paper is based on the dominant pole concept. Compensation zeroes are strategically placed near or within the acceptable closed loop pole region (See Figure 1.1). The gain of the system is then determined such that the closed loop poles remain within the acceptable region in the s-plane, despite the large plant parameter variations. "Far-off" poles (and zeroes) are then assigned in such a manner as to allow the loop transmission to decrease as fast as possible without violating the system time domain specifications. The problem of the placement of these "far-off" poles and zeroes, excluding the nearest "far-off" pole placed on the real axis, is not considered in this paper. Horowitz[4] has presented a method for the placement of these "far-off" poles and zeroes.

1.4 Scope of Work and Terminology

The plant parameter variation and acceptable dominant closed loop pole region are somewhat similar to those encountered in flight control.

Chapter II considers the problem of variation in the plant gain only. Plant pole cancellation and replacement is anticipated. The plant transfer function P(s) is assumed to be of the form

$$P(s) = \frac{k}{s(s^2+S_p s+P_p)} \qquad (1.1)$$

where: k = gain factor of the plant which may vary between k_{min} and k_{max};

S_p and P_p are fixed parameters that determine the position of the plant poles.

For complex plant poles located at $\sigma_p \pm j\omega_p$, S_p and P_p are given by

$$S_p = -2\sigma_p \qquad (1.2)$$

$$P_p = \sigma_p^2+\omega_p^2 \qquad (1.3)$$

The loop transmission $L_d(s)$ is assumed to have the form

$$L_d(s) = \frac{kK(s^2+S_o s+P_o)}{s(s^2+S_\ell s+P_\ell)(s+P_1)} \triangleq \frac{kKn_d(s)}{d_d(s)} \qquad (1.4)$$

where: K = fixed gain added to the system;

S_o and P_o are fixed parameters that determine the position of the compensation zeroes;

S_ℓ and P_ℓ are fixed parameters that determine the position of the plant replacement poles; $-P_1$ is the location of the nearest "far-off" open loop pole on the real axis.

For complex compensation zeroes located at $\sigma_z \pm j\omega_z$, S_o and P_o are given by

$$S_o = -2\sigma_z \tag{1.5}$$

$$P_o = \sigma_z^2 + \omega_z^2 \tag{1.6}$$

For plant replacement poles located on the real axis at r_1 and r_2, S_ℓ and P_ℓ are given by

$$S_\ell = -(r_1 + r_2) \tag{1.7}$$

$$P_\ell = r_1 r_2 \tag{1.8}$$

The closed-loop transfer function $T_d(s)$ is assumed to have the form

$$T_d(s) = \frac{P_r p_{f_1} p_{f_2}}{(s^2 + S_r s + P_r)(s + p_{f_1})(s + p_{f_2})} \triangleq \frac{P_r p_{f_1} p_{f_2}}{D_d(s)} \tag{1.9}$$

where: $-p_{f_1}$, $-p_{f_2}$ are the positions of the non-dominant closed loop poles which may be real or complex conjugate;

S_r and P_r are parameters that determine the position of the dominant closed loop poles.

For complex dominant closed loop poles located at $\sigma_d \pm j\omega_d$, S_r and P_r are given by

$$S_r = -2\sigma_d \tag{1.10}$$

$$P_r = \sigma_d^2 + \omega_d^2 \tag{1.11}$$

Chapter III considers the problem of simultaneous plant gain and plant pole variation. The plant transfer function has the same form as in Eq. 1.1 but S_p and P_p are now slowly varying or unknown parameters. Since plant pole cancellation is not practical here, the loop transmission has the form

$$L_d(s) = \frac{kk(s^2+S_o+P_o)}{s(s^2+S_I \quad +P_p)(s+p_1)} \triangleq \frac{kkn_d(s)}{d_d(s)} \tag{1.12}$$

The closed loop transfer function is of the same form as in Eq. 1.9.

Chapter IV considers the problem of simultaneous plant gain, plant pole and plant zero variation.

The plant transfer function is assumed to be of the form

$$P(s) = \frac{k(s+z)}{s(s^2+S_p s+P_p)} \qquad (1.13)$$

where: $-z$ is the position of the drifting zero on the real axis.

The loop transmission has the form

$$L_d(s) = \frac{kh(s^2+S_o s+P_o)(s+z)}{s(s^2+S_p s+P_p)(s+P_z)} \triangleq \frac{kKn_d(s)}{d_d(s)} \qquad (1.14)$$

where: $-P_z$ is the position of the fixed pole used to partially cancel the effect of the drifting zero.

The closed loop transfer function is

$$T_d(s) = \frac{P_r p_{f_1} p_{c_z}/z \quad (s+z)}{(s^2+S_r s+P_r)(s+p_{f_1})(s+p_{c_z})}$$

$$= \frac{P_r p_{f_1} p_{c_z}/z \quad (s+z)}{D_d(s)} \qquad (1.15)$$

where: $-p_{c_z}$ is the position of the closed loop pole near the drifting zero z on the real axis;

$-p_{f_1}$ is the position of the "far-off" closed loop pole on the real axis.

Chapters II and III are extensions of work by Horowitz[4] in that the effect of the "far-off" pole P_1 has been included while Chapter IV is completely original.

Appendix A presents various convergence procedures for factoring polynomials on a digital computor. Appendix B presents a geometric proof of a design procedure used in Chapter II.

CHAPTER II

DESIGN FOR PLANT GAIN VARIATION ONLY

2.1 Plant Pole Cancellation

This chapter treats the case when the plant pole variation is "sufficiently small" such that cancellation of the plant poles is valid providing, of course, that they are not located in the right half plane.[5] A discussion on what constitutes "sufficiently small" variation in the plant poles is given in (4). The saving in the gain-bandwidth product of the loop transmission, obtained by cancelling the plant poles and replacing them with poles nearer the desired closed loop pole region, may be quite substantial, especially if the acceptable closed loop pole region is far removed from the original plant poles.

2.2 Design Philosophy

The design philosophy in the case of plant gain variation only is to first cancel the existing plant poles and replace them with poles nearer the desired closed loop pole region. These poles will be near or on the boundary of the acceptable closed loop pole region at minimum plant gain, $k = k_{min}$. Compensation

zeroes are then located so that the dominant closed loop poles lie within the acceptable region despite the variat' 'ns in plant gain factor k. The problem is depicted pictorially in Fig. 2.1.

2.3 Design Equations

The expression for the loop transmission $L_d(s)$ is from Eq. 1.4

$$L_d(s) = \frac{kh(s^2+S_o s+P_o)}{s(s^2+S_\ell s+P_\ell)(s+P_1)} \triangleq \frac{khn_d(s)}{d_d(s)} \qquad (2.1)$$

and the expression for the system transmission $T_d(s)$ is from Eq. 1.9

$$T_d(s) = \frac{P_r p_{f_1} p_{f_2}}{(s^2+S_r s+P_r)(s+p_{f_1})(s+p_{f_2})} \triangleq \frac{P_r p_{f_1} p_{f_2}}{D_d(s)} \qquad (2.2)$$

The characteristic equation of the system is then

$$D_d(s) = d_d(s)+khn_d(s) \qquad (2.3)$$

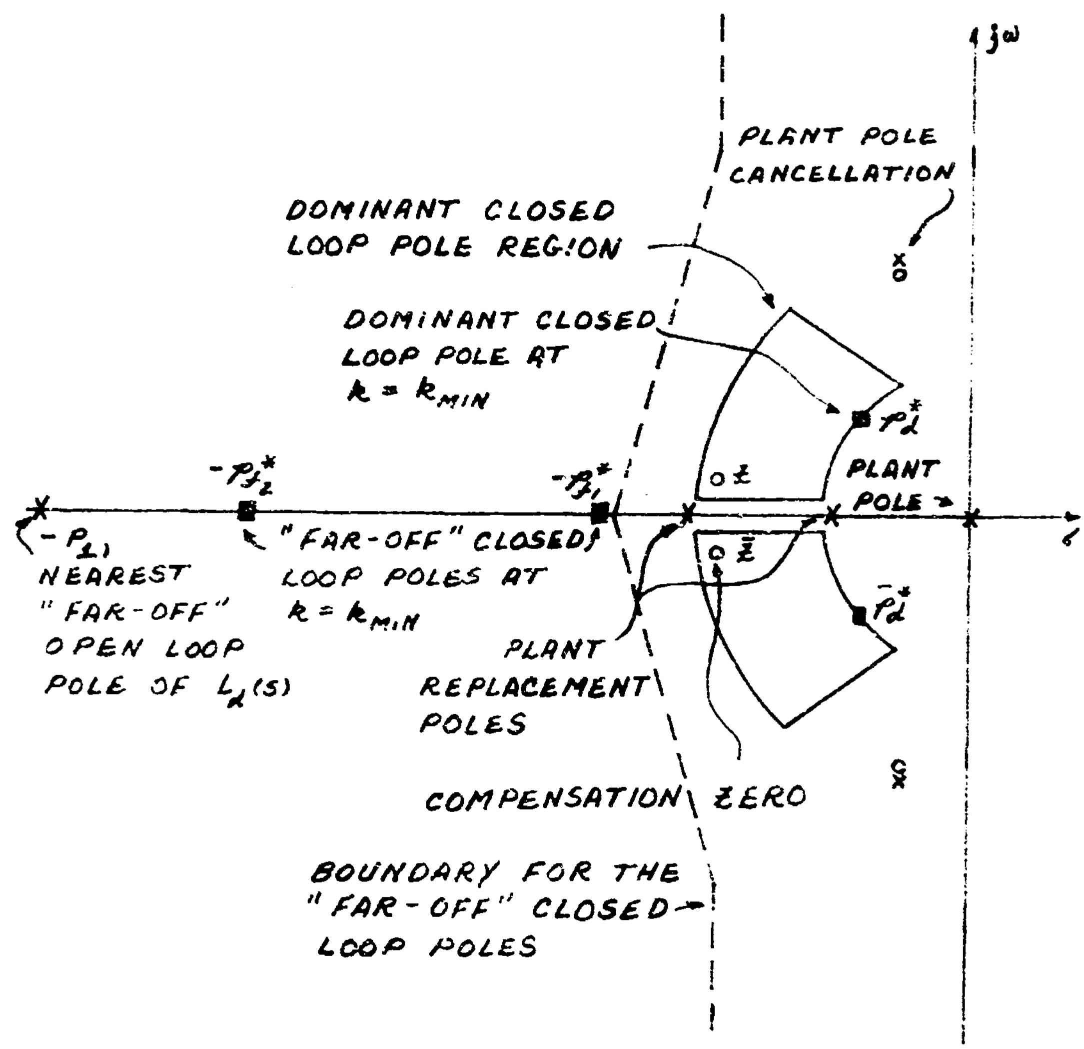
Equating the zero degree coefficients in Eq. 2.3 gives

$$khP_o = p_{f_1} p_{f_2} P_r \qquad (2.4)$$

FIG. 2.1 PROBLEM OF PLANT GAIN VARIATION ONLY

$$L_d(s) = \frac{kK(s^2+S_o s+P_o)}{s(s^2+S_l s+P_l)(s+P_1)}$$

$$T_d(s) = \frac{P_r\, p_{f_1}\, p_{f_2}}{(s^2+S_r s+P_r)(s+p_{f_1})(s+p_{f_2})}$$



Let P_r^*, $p_{f_1}^*$ and $p_{f_2}^*$ denote P_r, p_{f_1} and p_{f_2} at $k = k_{min}$. Also define

$$K_1 \triangleq k_{min} K = \frac{P_r^* p_{f_1}^* p_{f_2}^*}{P_o} \qquad (2.5)$$

It is very desirable to minimize the value of K_1, which is the system gain necessary to bring the root loci to the acceptable region of the dominant closed loop poles. Also, the high frequency asymptote of $L_d(s)$ (Eq. 2.1) is K_1/s^2, which is an important factor in determining the effect of internal noise at the plant input.[6] Large K_1 increases the possibility of plant saturation by internal high frequency noise. The next section deals with the choice of the parameters in Eq. 2.5 in which the minimization of K_1 is the prime objective.

2.4 Choice of Design Parameters to Minimize System Gain

The choice of the position of the dominant closed loop poles at $k = k_{min}$ depends somewhat on the shape of the acceptable region for the closed loop poles. Let the position of the dominant closed loop poles at $k = k_{min}$ be denoted by

$$p_d^*, \bar{p}_d^* = \sigma_d^* \pm j\omega_d^* \qquad (2.6)$$

Note that

$$P_r^* = |p_d^*|^2 = (\sigma_d^*)^2+(\omega_d^*)^2 \quad \text{(See Fig. 2.2)} \quad (2.7)$$

It is desirable to minimize $|p_d^*|$ and thereby P_r^*, since, from Eq. 2.5 this tends to minimize k_1. This implies that p_d^* should be located on or very close to the boundary of the acceptable region for the dominant closed loop poles.

The next problem is to choose the values for $p_{f_1}^*$ and $p_{f_2}^*$, the non-dominant closed loop poles at $k = k_{min}$. These closed loop poles will lie on the real axis for small values of gain since P_1 is assumed to be on the real axis. The values of $p_{f_1}^*$ and $p_{f_2}^*$ will have to be determined from considerations of the time domain specifications of the system transmission. They should be chosen as close in as possible, since from Eq. 2.5, this will tend to minimize k_1 but if they are too close to the origin, the system response can no longer be characterized by the dominant pole pair. The latter consideration determines the minimum distance (from the origin) of these poles. Henceforth, it is assumed that $p_{f_1}^*$ and $p_{f_2}^*$ are known.

The position of the compensation zeroes Z and $\overline{Z}$ is now considered. Denote the position of the zeroes Z, $\overline{Z}$ as

$$Z,\overline{Z} = \sigma_z \pm j\omega_z \qquad (2.8)$$

FIG. 2.2 THE EFFECT OF THE DOMINANT CLOSED LOOP POLE AND ZERO LOCATION ON K_1

$$K_1 = k_{MIN} K = \frac{p_{f_1}^* \, p_{f_2}^* \, P_r^*}{P_0}$$

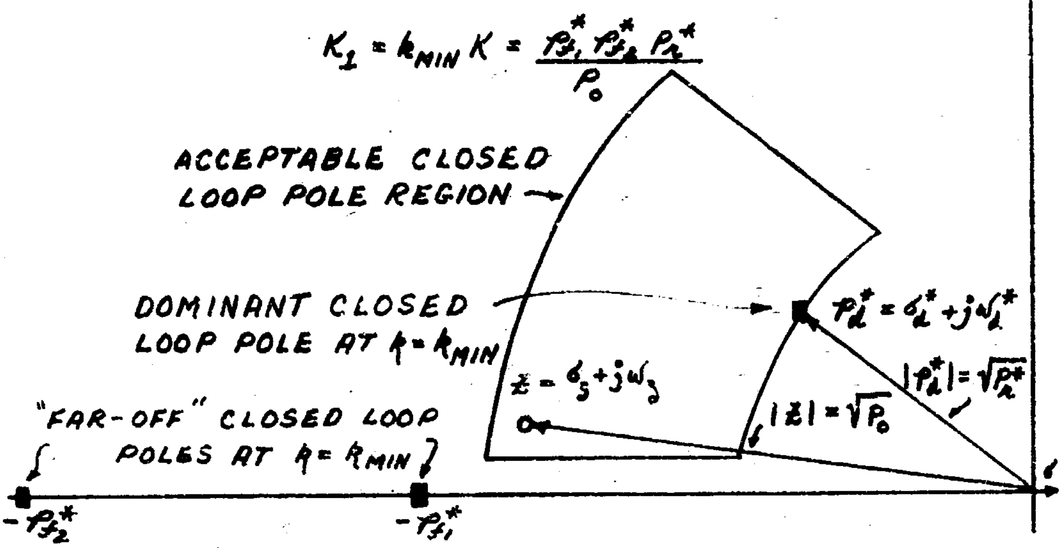
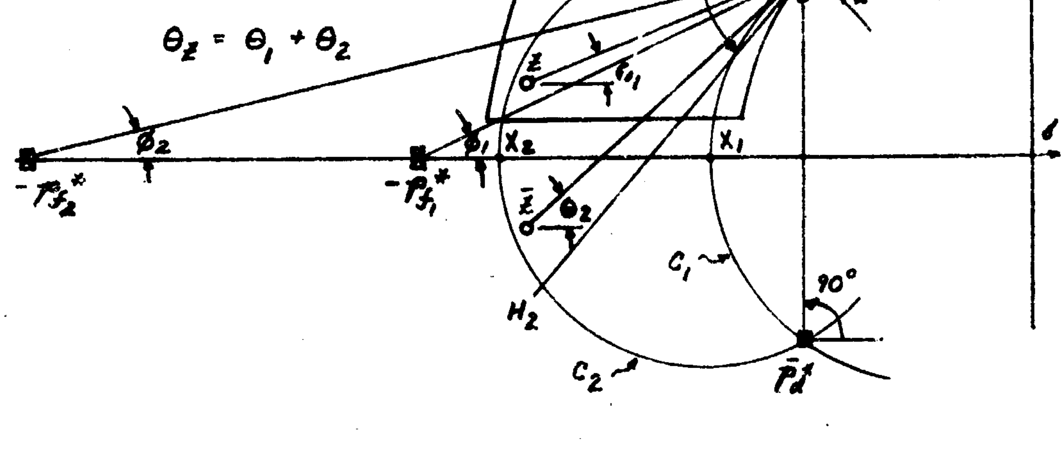


FIG. 2.3 LOCATION OF THE COMPENSATION ZEROES FROM THE ANGLE OF DEPARTURE OF THE ROOT LOCUS FROM p_d^*



Note that

$$P_o = |Z|^2 = \sigma_z^2 + \omega_z^2 \quad \text{(See Fig. 2.2)} \qquad (2.9)$$

Therefore $|Z|$ should be made as large as possible to maximize P_o which, from Eq. 2.5, will tend to minimize K_1. If the variation in plant gain is very large, the dominant closed loop poles will, at $k = k_{max}$, be very close to the compensation zeroes. This means that these zeroes must be located near the boundary of the acceptable closed loop pole region.

2.5 Positioning of Dominant Closed Loop Poles and Compensation Zeroes

The method used in this paper to fix the position of the compensation zeroes is the same as in reference (4). This method is to demand that the angle of departure of the root locus from the dominant pole p_d^* for $k > k_{min}$ be within a prescribed sector, given in Fig. 2.3 as $H_1 p_d^* H_2$. The choice of this sector is somewhat arbitrary but should be fairly general and easily applied to different acceptable regions of dominant closed loop poles.

In Fig. 2.3, the angle of departure, ψ_d, of the root locus from p_d^* is given by

$$\psi_d = 180° - (\Phi_1 + \Phi_2 + 90° - \theta_Z) \qquad (2.10)$$

where: $\Phi_1 = \angle - p^*_{f_1} p^*_d$

$\Phi_2 = \angle - p^*_{f_2} p^*_d$

$90° = \angle \bar{p}^*_d p^*_d$

$\theta_1 = \angle Z p^*_d$

$\theta_2 = \angle \bar{Z} p^*_d$

$\theta_Z = \theta_1 + \theta_2$

Solving Eq. 2.10 for θ_Z gives

$$\theta_Z = \psi_d + (\Phi_1 + \Phi_2 + 90°) - 180° \qquad (2.11)$$

Denote the extreme values of the departure angle ψ_d, which lie within the sector $H_1 p^*_d H_2$, as

$$\psi_{dmin} \leq \psi_d \leq \psi_{dmax} \qquad (2.12)$$

The extreme values of θ_Z are then

$$\theta_{Zmin} = 90° + \Phi_1 + \Phi_2 + \psi_{dmin} - 180° \qquad (2.13)$$

$$\theta_{Zmax} = 90° + \Phi_1 + \Phi_2 + \psi_{dmax} - 180° \qquad (2.14)$$

The locus of zero positions, Z, $\bar{Z}$, such that θ_Z is a constant, is an arc of a circle drawn through the

points p_d^*, $\overline{p}_d^*$ and a third point X on the real axis defined by the equation

$$\angle\, Xp_d^* = \theta_Z/2 \tag{2.15}$$

The proof of this statement is given in Appendix B.

The design procedure is, then, to locate two points X_1 and X_2 on the real axis corresponding to θ_{Zmin} and θ_{Zmax} in Eq. 2.15. Circular arcs c_1 and c_2 are drawn through the points $p_d^* X_1 \overline{p}_d^*$ and $p_d^* X_2 \overline{p}_d^*$, as shown in Fig. 2.3. Locating the compensation zeroes between the arcs c_1 and c_2 will then insure that the angle of departure of the root locus from the dominant pole p_d^* is within the specified sector $H_1 p_d^* H_2$.

When k_{max} is much greater than k_{min}, the closed loop poles will, at $k = k_{max}$, be very close to the compensation zeroes. Therefore, in order to insure that the root locus remains in the acceptable region as k approaches k_{max}, the angle of entry of the locus into the complex zero should be checked. The angle of entry ϕ_e, is given by

$$\phi_e = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 90° - 180° \tag{2.16}$$

where: $\alpha_1 = \angle -p_{f_1}^* Z$

$\alpha_2 = \angle -p_{f_2}^* Z$

$$\alpha_3 = \angle p_d^* Z$$

$$\alpha_4 = \angle \bar{p}_d^* Z$$

$$90° = \angle \bar{Z}\ Z$$

These angles are shown in Fig. 2.4.

The value K_1 may be computed from Eq. 2.5 once the zeroes have been located. It should be noted that if k_{max} is not much greater than k_{min}, the zeroes may not have to be located within the acceptable closed loop pole region.

2.6 Open Loop Poles of $L_d(s)$

The last step in the design procedure is to locate the open loop poles of $L_d(s)$. From Eq. 2.3, the expression for $D_d^*(s)$ is

$$D_d^*(s) = d_d(s)+K_1 n_d(s)$$

Therefore

$$d_d(s) = D_d^*(s)-K_1 n_d(s) \tag{2.17}$$

or

$$d_d(s) = (s^2+S_r^* s+P_r^*)(s+p_{f_1}^*)(s+p_{f_2}^*)-K_1(s^2+S_o s+P_o) \tag{2.18}$$

FIG. 2.4 CALCULATION OF THE ANGLE OF ENTRY OF THE ROOT LOCUS INTO THE COMPENSATION ZERO z

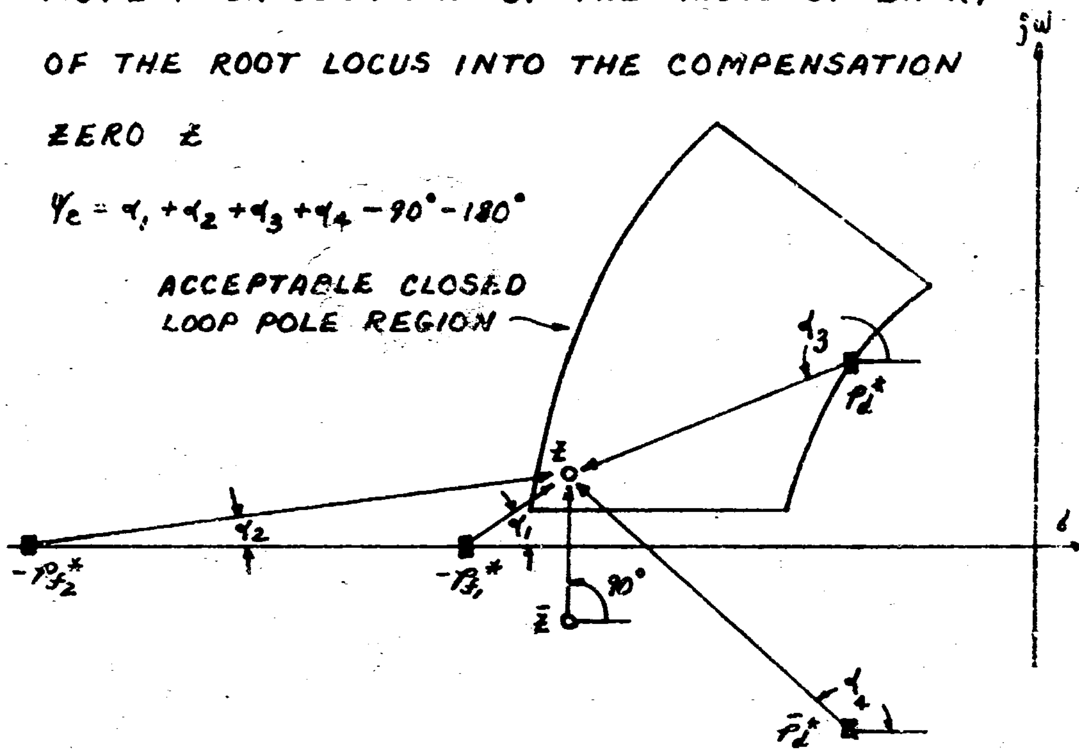
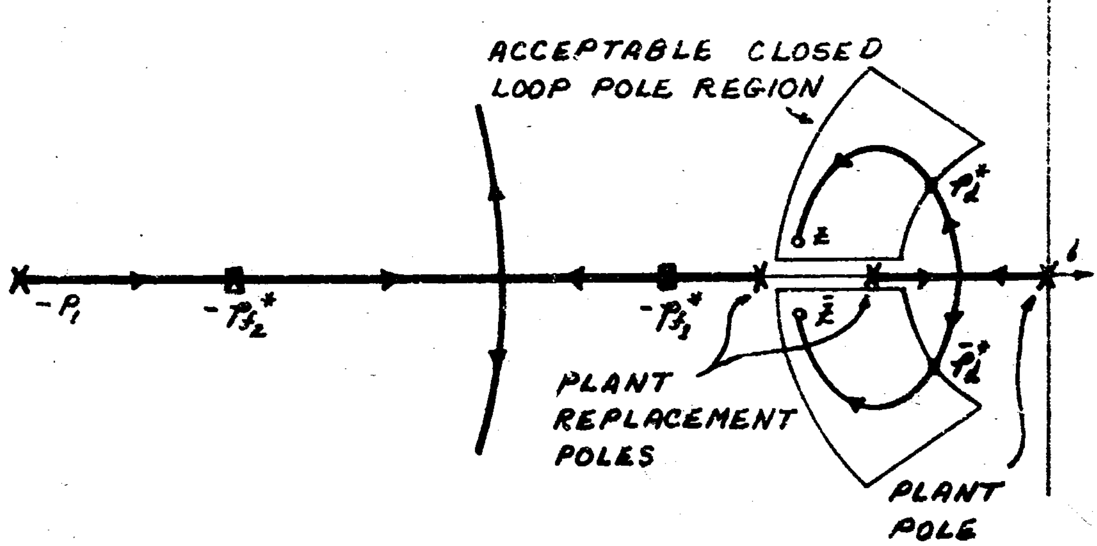


FIG. 2.5 ANTICIPATED ROOT LOCUS



$$=s\{s^3+(p^*_{f_1}+p^*_{f_2}+S^*_r)s^2+(p^*_{f_1}p^*_{f_2}+S^*_r(p^*_{f_1}+p^*_{f_2})+P^*_r-K_1)s$$

$$+(S^*_r p^*_{f_1}p^*_{f_2}+P^*_r(p^*_{f_1}+p^*_{f_2})-K_1S_o)\} \tag{2.19}$$

$$= s(s^2+S_\ell s+P_\ell)(s+P_1) \tag{2.20}$$

The roots of $(s^2+S_\ell s+P_\ell)$ may be complex or real depending on the location of the acceptable closed loop pole region and the value of K_1. The anticipated root locus, for a design of this type, for the given acceptable closed loop pole region, is shown in Fig. 2.5.

2.7 Design Example

The following is an application of the design procedure for gain variation only.

The design equations are

$$T^*_d(s) = \frac{p^*_{f_1}p^*_{f_2}P^*_r}{(s^2+S^*_r s+P^*_r)(s+p^*_{f_1})(s+p^*_{f_2})}$$

$$L^*_d(s) = \frac{k_{min}K(s^2+S_o s+P_o)}{s(s^2+S_\ell s+P_\ell)(s+P_1)}$$

where $T^*_d(s)$ and $L^*_d(s)$ are the system transmission and loop transmission respectively, at $k = k_{min}$. The plant transfer function is assumed to be of the form

$$P(s) = \frac{k}{s(s^2+S_p s+P_p)}$$

where k, the plant gain factor, may vary from $k_{min}=1$ to $k_{max}=1000$. The complex plant poles are assumed to be fixed and have been cancelled by zeroes placed near them.

The closed loop pole values $-p^*_{f_1}$ and $-p^*_{f_2}$ are assumed to be -10 and -15 respectively. These choices for $p^*_{f_1}$ and $p^*_{f_2}$ are purely arbitrary other than that they must lie to the left of the boundary shown in Fig. 2.6. This report does not show how these values of $p^*_{f_1}$ and $p^*_{f_2}$ are obtained but how the design procedes once they are known. The acceptable region for the dominant closed loop poles, as well as other pertinent quantities for this design example, are shown in Fig. 2.6.

As a first choice for p^*_d, the dominant closed loop pole at $k = k_{min}$, let

$$p^*_d = -3+j3$$

since this is close to the minimum value of P_r for this acceptable dominant closed loop pole region. The angle of departure of the root locus from the dominant pole p^*_d must be within the sector $H_1 p^*_d H_2$, i.e.

$$140° \le \psi_d \le 230°$$

FIG. 2.6 DESIGN EXAMPLE FOR PLANT GAIN VARIATION ONLY

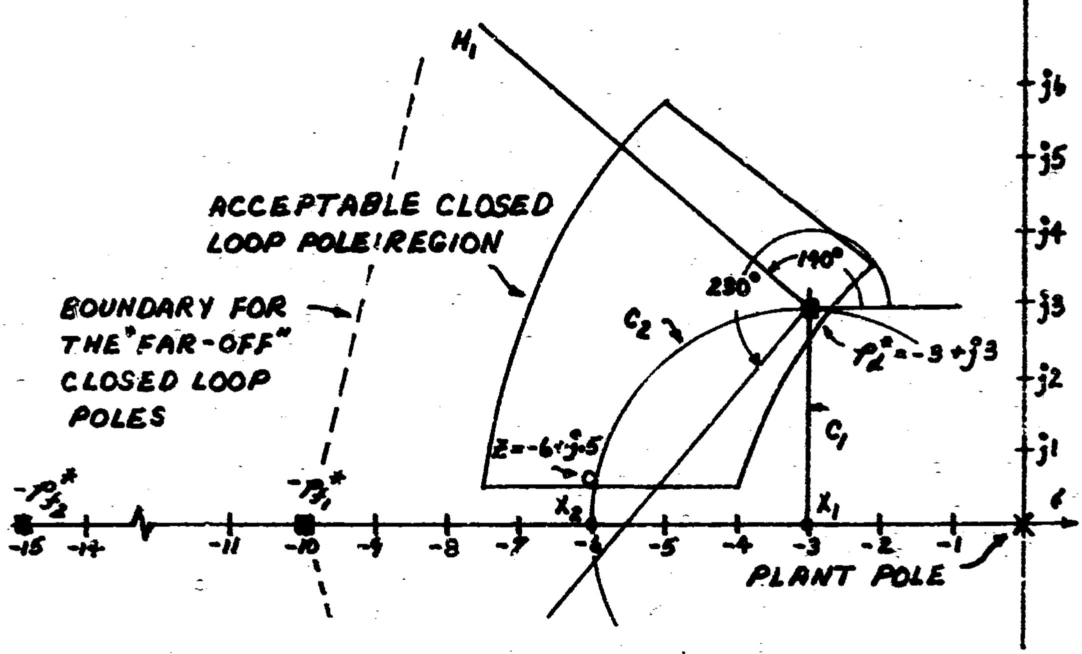
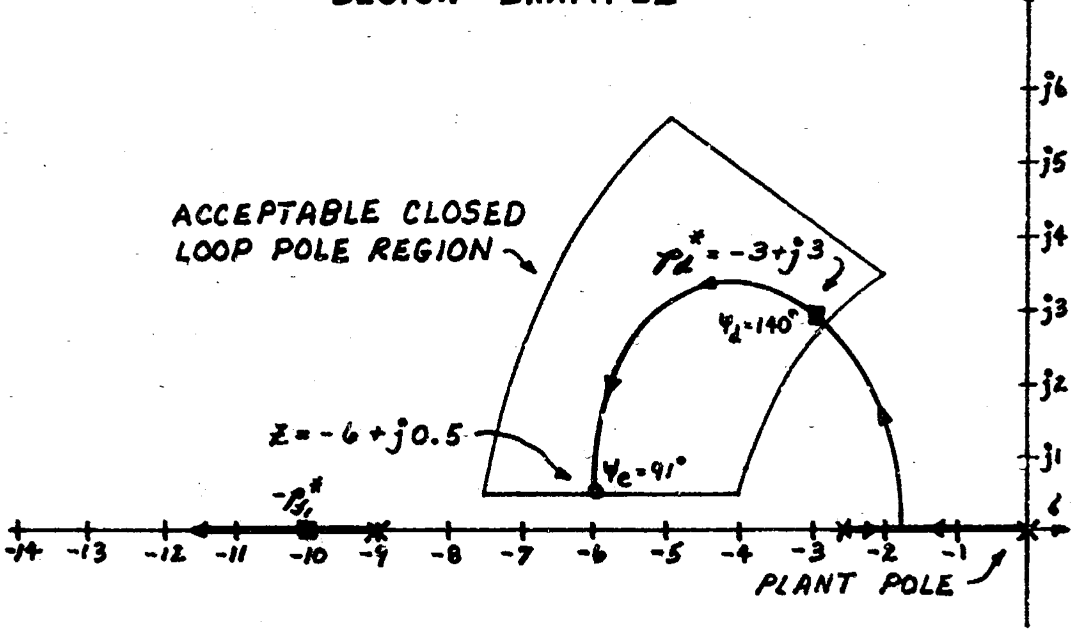


FIG. 2.7 APPROXIMATE ROOT LOCUS FOR THE DESIGN EXAMPLE



The angle of departure of the root locus from the dominant pole p_d^* is given by

$$\psi_d = 180° - (\Phi_1 + \Phi_2 + 90° - \theta_Z)$$

where: $\Phi_1 = \angle -p_{f_1}^* p_d^* = \mathrm{Tan}^{-1} \frac{3}{7} = 25.4°$

$\Phi_2 = \angle -p_{f_2}^* p_d^* = \mathrm{Tan}^{-1} \frac{3}{12} = 14.0°$

$\theta_Z = \angle Z p_d^* + \angle \bar{Z} p_d^*$

From Eqs. 2.13 and 2.14, the values of θ_{Zmax} and θ_{Zmin} are given by

$$\begin{aligned}\theta_{Zmax} &= \psi_{dmax} - 90° + \Phi_1 + \Phi_2 \\ &= 230° - 90° + 25.4° + 14.0° \\ &= 179.4°\end{aligned}$$

$$\begin{aligned}\theta_{Zmin} &= \psi_{dmin} - 90° + \Phi_1 + \Phi_2 \\ &= 140° - 90° + 25.4° + 14.0° \\ &= 89.4°\end{aligned}$$

Next, two points on the real axis X_1, X_2 are determined such that

$$\angle X_1 p_d^* = \frac{\theta_{Zmax}}{2} \cong 90°$$

$$\angle X_2 p_d^* = \frac{\theta_{Zmin}}{2} \cong 45°$$

Two arcs are now drawn through the points $p_d^* X_1 \bar{p}_d^*$ and $p_d^* X_2 \bar{p}_d^*$ (See Fig. 2.6). The construction begins by locating a point on the real axis that is equidistant from the points p_d^*, X and $\bar{p}_d^*$. Using this point on the real axis as the center of a circle, circular arcs are drawn through the points p_d, X and $\bar{p}_d^*$ with a compass.

As a first choice for the position of the compensation zeroes Z and $\bar{Z}$, let

$$Z, \bar{Z} = -6 \pm j0.5$$

This position for the compensation zeroes will tend to maximize P_o for the circular arcs c_1 and c_2 shown in Fig. 2.6. Large P_o from Eq. 2.5 will mean a smaller value for k_1 which is the object of this design procedure.

The expression for $n_d(s)$ is

$$\begin{aligned} n_d(s) &= (s-Z)(s+\bar{Z}) \\ &= (s+6+j0.5)(s+6-j0.5) \\ &= s^2+12s+36.25 \end{aligned}$$

The value of k_1 may be computed from Eq. 2.5, i.e.

$$\begin{aligned} k_1 &= k_{min}k = 1 \cdot k \\ &= \frac{P_r^* p_{f_1}^* p_{f_2}^*}{P_o} \\ &= \frac{(18)(10)(15)}{36.25} = 74.5 \end{aligned}$$

From Eqs. 2.17 and 2.19, the expression for $d_d(s)$ is

$$d_d(s) = D_d^*(s) - K_1 n_d(s)$$

or

$$d_d(s) = s\{s^3 + (p_{f_1}^* + p_{f_2}^* + S_r^*)s^2 + (p_{f_1}^* p_{f_2}^* + S_r^*(p_{f_1}^* + p_{f_2}^*) + P_r - K_1)s$$

$$+ (S_r^* p_{f_1}^* p_{f_2}^* + P_r(p_{f_1}^* + p_{f_2}^*) - K_1 S_o)\}$$

$$= s\{s^3 (10+15+6)s^2 + (150+6(25)+18-74.5)s$$

$$+ (6(150)+18(25)-12(74.5))\}$$

$$= s(s^3 + 31s^2 + 243.5s + 456)$$

To obtain the open loop poles of $L_d(s)$, this equation must be factored. Using the methods in Appendix A, this equation is factored into the following open loop poles

$$d_d(s) = s(s^2 + 28.4s + 168)(s + 2.65)$$

$$= s(s+2.65)(s+9.00)(s+19.44)$$

from which

$$P_1 = 19.44$$

$$(s^2 + S_\ell s + P_\ell) = (s+2.65)(s+9.00)$$

$$= s^2 + 11.65s + 23.85$$

The angle of departure ψ_d of the root locus from the dominant pole p_d^* is

$$\psi_d \quad 180° - (\Phi_1 + \Phi_2 + 90° - \theta_Z)$$

$$\theta_Z = \angle Z p_d^* + \angle \bar{Z} p_d^*$$

$$= \mathrm{Tan}^{-1}\frac{2.5}{3} + \mathrm{Tan}^{-1}\frac{3.5}{3} = 39.8° + 49.3°$$

$$= 89.1°$$

$$\text{Therefore } \psi_d = 180° - (25.4° + 14.0° + 90° - 89.1°)$$

$$= 139.7°$$

which is satisfactory since it is very close to the minimum value of 140°. The angle of entry of the root locus into the zeroes is given by Eq. 2.15, i.e.

$$\psi_e = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 90° - 180°$$

$$= \angle -p_{f_1}^* Z + \angle -p_{f_2}^* Z + \angle p_d^* Z + \angle \bar{p}_d^* Z - 270°$$

$$= \mathrm{Tan}^{-1}\frac{.5}{4} + \mathrm{Tan}^{-1}\frac{.5}{9} + \mathrm{Tan}^{-1}\frac{-2.5}{-3} + \mathrm{Tan}^{-1}\frac{3.5}{-3} - 270°$$

$$= 7.1° + 3.2° + 219.8° + 130.7° - 270°$$

$$= 90.8°$$

which is satisfactory for this design (see Fig. 2.6). For this particular placement of the zeroes, an angle of entry between 20° and 160° would probably be satisfactory. The approximate root locus for this design is shown in Fig. 2.7.

2.8 Improvement in Design Example

The first design could be improved. If the compensation zeroes could be moved further to the left, it would, from Eq. 2.5, decrease the value of K_1. With this as the objective, let the second choice for

p_d^* be $\qquad p_d^* = -3.5+j2.0$

The angle of departure ψ_d is constrained to be within the sector $H_1 p_d^* H_2$ defined by the equation

$$110° \le \psi_d \le 230°$$

Quantities pertinent to this design are shown in Fig. 2.8.

The angles Ψ_1 and Φ_2 are

$$\Psi_1 = \text{Tan}^{-1}\frac{2}{6.5} = \text{Tan}^{-1}0.308 = 17.2°$$

$$\Phi_2 = \text{Tan}^{-1}\frac{2}{11.5} = \text{Tan}^{-1}0.174 = 9.85°$$

The same method as before will be used to fix the position of the compensation zeroes. The angle of departure of the root locus from p_d^* is

$$\psi_d = 180°-(\Psi_1+\Phi_2+90°-\theta_Z)$$

$$= 90°-17.2°-9.85°+\theta_Z)$$

or $\qquad \theta_Z = \psi_d - 62.95°$

Therefore $\quad \theta_{Zmax} \cong 230°-63° = 167°$

$$\theta_{Zmin} \cong 110°-63° = 47°$$

Again two points on the real axis X_1, X_2 are determined such that

$$\angle X_1 p_d^* = 83.5°$$

$$\angle X_2 p_d^* = 23.5°$$

FIG. 2.8 IMPROVEMENT IN THE DESIGN FOR PLANT GAIN VARIATION ONLY

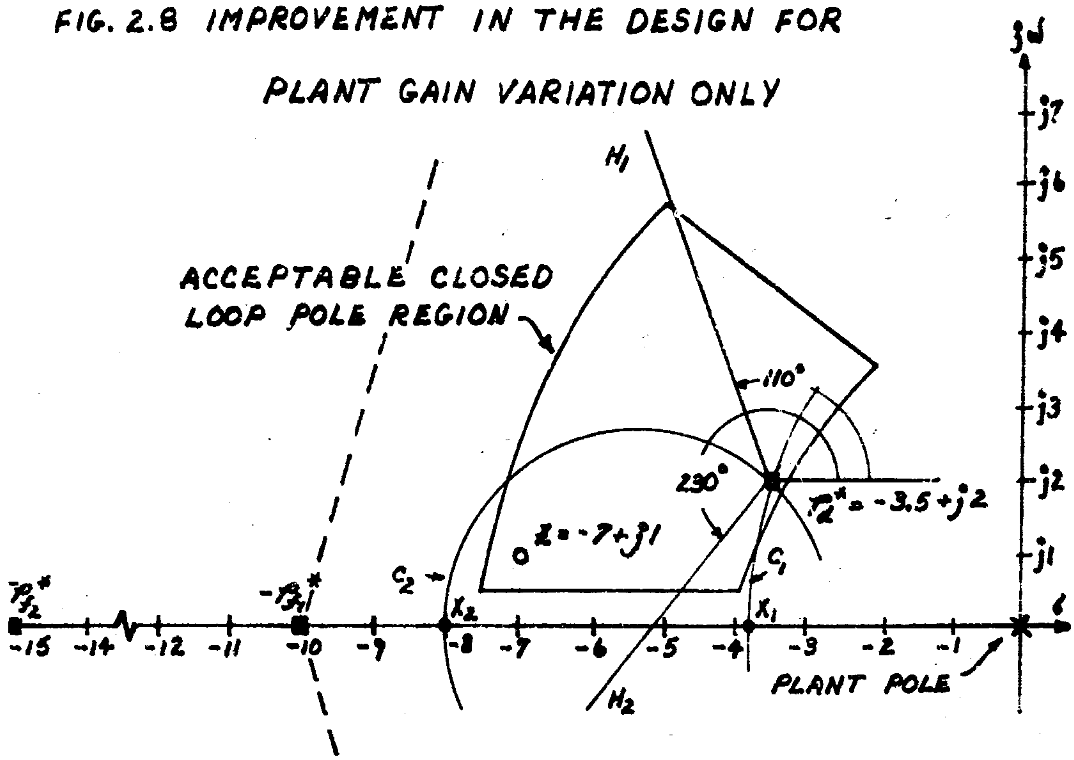
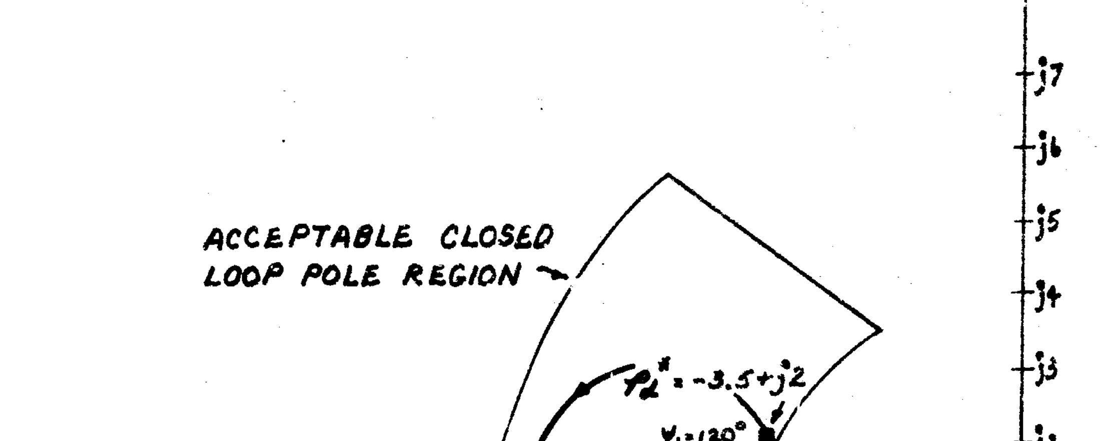


FIG. 2.9 APPROXIMATE ROOT LOCUS FOR THE SECOND DESIGN



Two circular arcs are now drawn through the points $p_d^* X_1 \overline{p}_d^*$ and $p_d^* X_2 \overline{p}_d^*$ as shown in Fig. 2.8.

The compensation zero positions are chosen as

$$Z, \overline{Z} = -7 \pm j1$$

The expression for $n_d(s)$ is then

$$n_d(s) = s^2 + 14s + 50$$

The value of K_1 from Eq. 2.5 is

$$K_1 = \frac{P_r^* p_{f_1}^* p_{f_2}^*}{P_o} = \frac{(16.23)(10)(15)}{50}$$

$$= 48.69$$

This is a reduction of about 4 db. from the K_1 of the previous design.

The angle of departure ϕ_d of the root locus from p_d^*, for this position of compensation zeroes, is 120°. The angle of entry ϕ_e of the root locus into the compensation zero Z is 84°. Both of these values are satisfactory for the given acceptable closed loop pole region (see Fig. 2.8). The reason that the angle of entry should be checked is as follows: If the compensation zeroes were placed in the extreme left hand corner of the acceptable closed loop pole region, an angle of entry greater than approximately 80° would be unsatisfactory. An angle of entry larger than 80° would probably indicate that the root locus would be

outside the acceptable closed loop pole region for some value of k between $k = k_{min}$ and $k = k_{max}$.

The expression for $d_d(s)$ is found from Eq. 2.19

$$d_d(s) = s\{s^3+(10+15+7)s^2+(150+7(25)+16.23-48.69)s$$
$$+(7(150)+16.23(25)-14(48.69))\}$$
$$= s\{s^3+32s^2+292.5s+773\}$$

Factoring this equation results in the following expression for $d_d(s)$

$$d_d(s) = (s+4.46)(s^2+27.7s+173)$$
$$= (s+4.46)(s+9.49)(s+18.21)$$

The root locus for this design is of the same form as the previous design and is shown in Fig. 2.9.

2.9 Summary of Design Procedure

The design procedure for variation in plant gain factor only is summarized below.

1. Cancel the plant poles.
2. Fix the position of the dominant closed loop poles at $k = k_{min}$.
3. Determine the values of the "far off" closed loop poles at $k = k_{min}$.
4. Determine the position of the compensation zeroes so that the root locus from $k = k_{min}$

to $k=k_{max}$ remains within the acceptable closed loop pole region.

5. Solve for the open loop poles of $L_d(s)$ from Eq. 2.20.

The next chapter in this paper considers variation in both the plant poles and plant gain factor.

CHAPTER III

PROBLEM OF SIMULTANEOUS PLANT GAIN AND PLANT POLE VARIATION

3.1 Problem Definition

In the case where the plant poles vary as well as the plant gain factor, plant pole cancellation is not feasible. In this case the system gain must be sufficiently high so that the dominant closed loop poles remain within their acceptable region despite the variations in the plant poles. The problem resolves into locating the compensation zeroes such that the system gain necessary to accomplish this is minimized.

3.2 Design Equations

The expressions for the dominant part of the loop transmission and system transmission given by Eqs. 1.12 and 1.9 are repeated below.

$$L_d(s) = \frac{kh(s^2+S_o s+P_o)}{s(s^2+S_p s+P_p)(s+P_1)} = \frac{kkn_d(s)}{d_d(s)}$$

$$T_d(s) = \frac{p_{f_1} p_{f_2} P_r}{(s^2+S_r s+P_r)(s+p_{f_1})(s+p_{f_2})} = \frac{p_{f_1} p_{f_2} P_r}{D_d(s)}$$

The plant transfer function is of the form

$$P(s) = \frac{k}{s(s^2+S_p s+P_p)}$$

The plant gain factor k varies from $k = k_{min}$ to $k = k_{max}$ and the plant poles may lie or "slowly vary" within the region shown in Fig. 3.1. The characteristic equation of the system $D_d(s)$ from Eq. 2.3 is

$$D_d(s) = d_d(s)+kKn_d(s)$$

$$\text{or } (s^2+S_r s+P_r)(s+p_{f_1})(s+p_{f_2}) = s(s^2+S_p s+P_p)(s+P_1) + kK(s^2+S_o s+P_o) \tag{3.1}$$

$$s^4+(S_r+p_{f_1}+p_{f_2})s^3+(p_{f_1}p_{f_2}+S_r(p_{f_1}+p_{f_2})+P_r)s^2+(S_r p_{f_1}p_{f_2} +P_r(p_{f_1}+p_{f_2}))s+P_r p_{f_1}p_{f_2} = s^4+(S_p+P_1)s^3+(S_p P_1+P_p+kK)s^2 +(P_p P_1+kKS_o)s+kKP_o \tag{3.2}$$

Equating the coefficients of Eq. 3.2 yields the following set of equations

$$S_r+p_{f_1}+p_{f_2} = S_p+P_1 \tag{3.3}$$

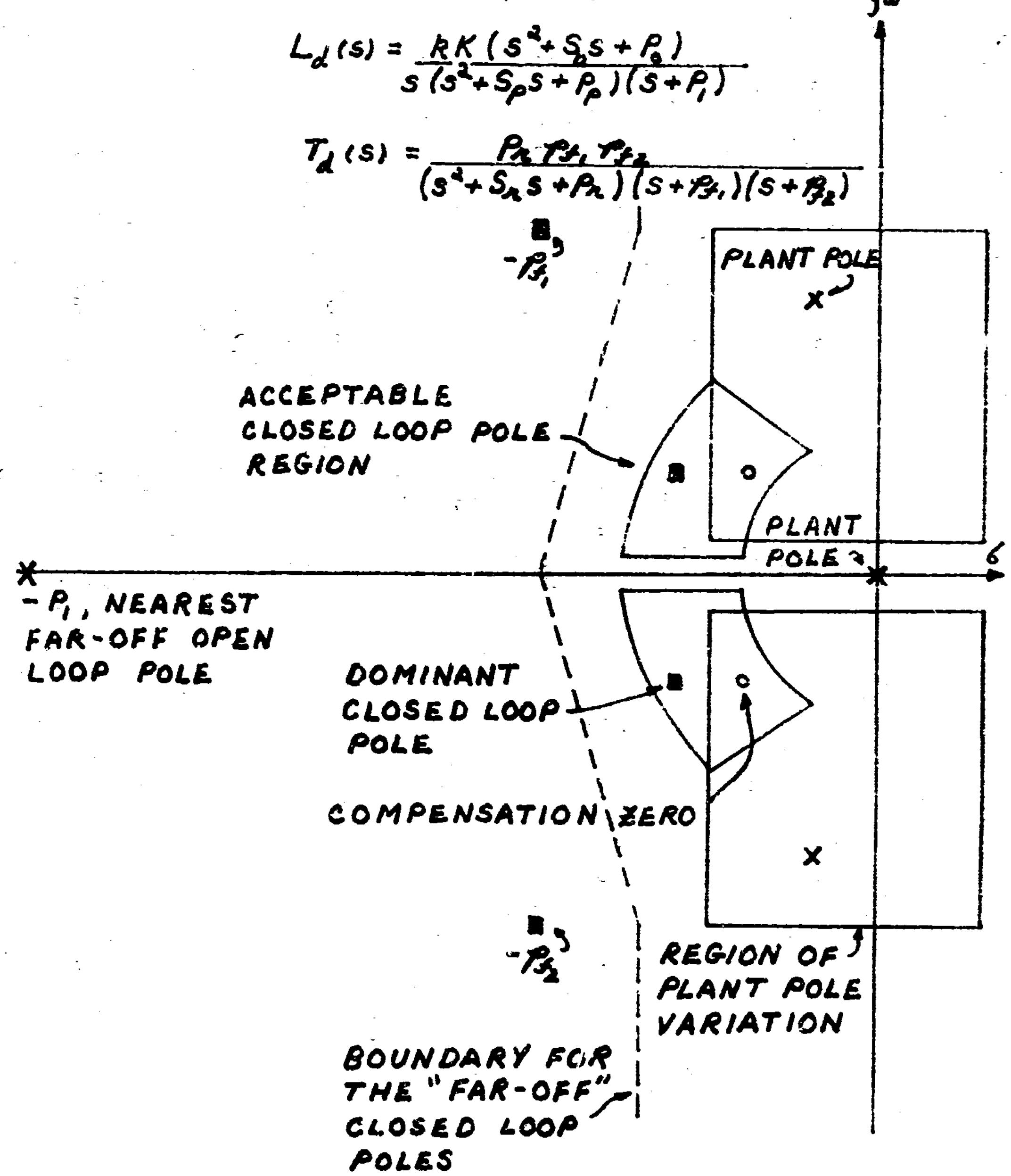
$$p_{f_1}p_{f_2}+S_r(p_{f_1}+p_{f_2})+P_r = S_p P_1+P_p+kK \tag{3.4}$$

FIG. 3.1 PROBLEM OF SIMULTANEOUS PLANT GAIN AND PLANT POLE VARIATION

$$P(s) = \frac{k}{s(s^2 + S_p s + P_p)}$$

$$L_d(s) = \frac{kK(s^2 + S_0 s + P_0)}{s(s^2 + S_p s + P_p)(s + P_1)}$$

$$T_d(s) = \frac{P_x p_{f_1} p_{f_2}}{(s^2 + S_x s + P_x)(s + p_{f_1})(s + p_{f_2})}$$



$$S_r p_{f_1} p_{f_2} + P_r(p_{f_1} + p_{f_2}) = P_p P_1 + k\Lambda S_o \qquad (3.5)$$

$$P_r p_{f_1} p_{f_2} = k\Lambda P_o \qquad (3.6)$$

The following substitutions in Eqs. 3.4 and 3.5 are made

$$p_{f_1} + p_{f_2} = S_p + P_1 - S_r \qquad (3.7)$$

$$p_{f_1} p_{f_2} = \frac{kKP_o}{P_r} \qquad (3.8)$$

$$\frac{k\Lambda P_o}{P_r} + S_r(S_p + P_1 - S_r) + P_r = S_p P_1 + P_p + k\Lambda \qquad (3.9)$$

$$\frac{S_r k\Lambda P_o}{P_r} + P_r(S_p + P_1 - S_r) = P_p P_1 + kKS_o \qquad (3.10)$$

Define the following quantites

$$\gamma \triangleq k\Lambda P_o \qquad (3.11)$$

$$X \triangleq S_p P_1 + P_p + k\Lambda \qquad (3.12)$$

$$Y \triangleq P_p P_1 + k\Lambda S_o \qquad (3.13)$$

Equations 3.9 and 3.10 are then

$$\frac{\gamma}{P_r} + S_r(S_p + P_1 - S_r) + P_r = X \qquad (3.14)$$

$$\frac{\gamma S_r}{P_r} + P_r(S_p + P_1 - S_r) = Y \qquad (3.15)$$

Considering the equations for X and Y, (Eqs. 3.12, 3.13), the variation in X, ΔX, and the variation in Y, ΔY, due to the variation in plant poles only, i.e., parameters S_p and P_p, can be expressed as

$$\Delta X = P_1 \Delta S_p + \Delta P_p \qquad (3.16)$$

$$\Delta Y = P_1 \Delta P_p \qquad (3.17)$$

The variation in plant gain factor will be considered later in the design.

3.3 Design Procedure

An outline of the design procedure that will be followed in this problem is as follows:

1.) Map the acceptable region for the dominant closed loop poles into the X,Y plane using Eqs. 3.14 and 3.15 for fixed values of the parameters γ, P_1 and S_p.

2.) Map the plant pole variation into the ΔX, ΔY plane using Eqs. 3.16 and 3.17 for fixed values of the parameter P_1.

3.) Compare the two mappings in (1,2) above. If the mapping of the plant pole variation in the ΔX, ΔY plane does not fit into the interior of the mapping of the acceptable dominant closed loop pole region, the mapping of the latter will have to be repeated,

using a larger value of γ.

4.) Solve for the values of kK, S_o and P_o using Eqs. 3.11, 3.12 and 3.13, where the values of X and Y are obtained from the positioning of the mapping of the plant pole variation in the interior of the mapping of the dominant closed loop pole region in the X, Y plane.

The next four sections in this chapter elaborate on these four steps in the design procedure.

3.4 Mapping of the Dominant Closed Loop Pole Region

The mapping of the acceptable dominant closed loop pole region into the X,Y plane involves the parameters γ, P_1 and S_p Large γ implies large gain, i.e., large kK, since P_o does not have a large range of values (See. Eq. 3.11). An approximation for the value of γ can be obtained from Eq. 3.6, i.e.

$$\gamma = p_{f_1} p_{f_2} P_r \tag{3.18}$$

The maximum value of P_r can be found from the acceptable r gion for the dominant closed loop poles. Denoting the value of the dominant closed loop pole by $p_d = \sigma_d + j\omega_d$, the value of P_r is

$$P_r = |p_d|^2 = \sigma_d^2 + \omega_d^2 \tag{3.19}$$

The values of p_{f_1} and p_{f_2} can be roughly approximated by considering the boundary for these closed loop poles shown in Fig. 3.1. These poles will be complex for large values of system gain.

P_1, the nearest "far off" open loop pole located on the real axis, should be chosen as close in as possible, since from Eq. 3.3, this will decrease the values of p_{f_1} and p_{f_2} which, from Eq. 3.8, will tend to decrease the value of fixed gain that must be added to the system. If P_1 is chosen too close in, though, the closed loop poles p_{f_1} and p_{f_2} may lie to the right of the vertical boundary shown in Fig. 3.1 violating the specifications of the problem.

The major problem in this mapping operation is the parameter S_p. Denoting the value of the plant pole as $p_{p\ell} = \sigma_p + j\omega_p$, the value of S_p is

$$S_p = -2\sigma_p \qquad (3.20)$$

In the point by point mapping of the acceptable region for the dominant closed loop poles, there is no criteria for determining what value of S_p to associate with a particular point on the boundary of the acceptable dominant closed loop pole region. This dilemma is resolved by considering the following argument: If the dominant closed loop poles are to lie within their

acceptable region despite the variation in the plant poles, the mapping of the dominant closed loop pole region can not be highly sensitive to the position of the plant poles and hence the value of S_p. The actual point by point mapping of the dominant closed loop pole region into the X,Y plane is performed using several different values of S_p for each value of γ and P_1. These different values of S_p should include the minimum and maximum values of S_p for the given region of plant pole variation as well as values in between. A median value of S_p is denoted by S_{pmed} in Fig. 3.2. The mapping of the acceptable closed loop pole region into the X,Y plane is depicted graphically in Fig. 3.2.

The values of p_{f_1} and p_{f_2}, the "far off" closed loop poles, are also of interest in the mapping of the dominant closed loop pole region. The values of p_{f_1} and p_{f_2} must lie to the left of the boundary shown in Fig.3.1. For each point on the boundary of the dominant closed loop pole region and given values of the parameters γ, P_1 and S_p, the values of p_{f_1} and p_{f_2} may be obtained as follows: From Eqs. 3.3 and 3.6

$$p_{f_1}+p_{f_2} = S_p+P_1-S_r \tag{3.21}$$

$$p_{f_1}p_{f_2} = \gamma/P_r \tag{3.22}$$

Define
$$p_{f_1}+p_{f_2} \triangleq S_f \tag{3.23}$$

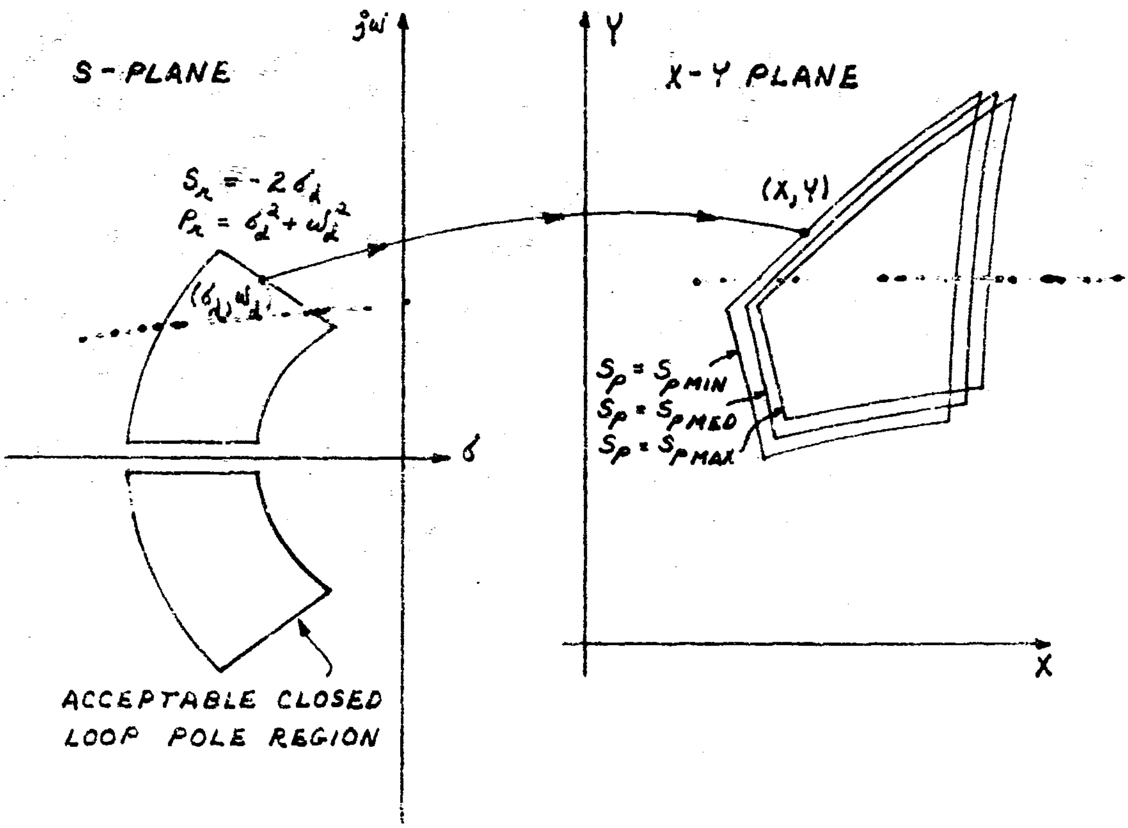
FIG. 3.2 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION IN THE S-PLANE INTO THE X-Y PLANE

MAPPING EQUATIONS

$$X = \frac{\gamma}{P_r} + S_r(S_p + P_1 - S_r) + P_r$$

$$Y = \frac{\gamma S_r}{P_r} + P_r(S_p + P_1 - S_r)$$

PARAMETERS γ, P_1, S_p



$$p_{f_1} p_{f_2} \triangleq P_f \tag{3.24}$$

Solving Eqs. 3.23 and 3.24 for p_{f_1} and p_{f_2} yields

$$p_{f_1} = \frac{S_f}{2} + \sqrt{(S_f/2)^2 - P_f} \tag{3.25}$$

$$p_{f_2} = \frac{S_f}{2} - \sqrt{(S_f/2)^2 - P_f} \tag{3.26}$$

If the values of p_{f_1} and p_{f_2} fall to the right of the vertical boundary shown in Fig. 3.1, P_1 has been placed too far in. The values of p_{f_1} and p_{f_2} obtained during this mapping operation will not correspond exactly with those in the final design, since the boundary of the plant pole variation will not, in general, map exactly onto the boundary of the acceptable dominant closed loop pole region. Additional features of this mapping operation are covered in Section 3.8.

3.5 Mapping of the Plant Pole Variation

The only parameter in the mapping of the plant pole variation into the ΔX, ΔY plane is P_1 (See Eqs. 3.16 and 3.17). The value of P_1 used in the mapping must be the same as that used in the mapping of the dominant closed loop pole region (Eqs. 3.14, 3.15). The mapping of the plant pole variation is implemented by defining any point $(\sigma_{po}, \omega_{po})$ on the boundary of the plant pole variation. The nominal values of S_p and P_p

are then $$S_{po} = -2\sigma_{po} \quad (3.27)$$

$$P_{po} = \sigma_{po}^2+\omega_{po}^2 \quad (3.28)$$

Equations 3.16 and 3.17 may then be written as

$$\Delta X = P_1(S_p-S_{po})+(P_p-P_{po}) \quad (3.29)$$

$$\Delta Y = P_1(P_p-P_{po}) \quad (3.30)$$

The region of plant pole variation is then mapped point by point into the ΔX, ΔY plane, as shown in Fig. 3.3. It should be noted at this point that the shape and size of the mapping of the plant pole variation in the ΔX, ΔY plane is not dependent on the choice of the point $(\sigma_{po},\omega_{po})$ and hence on the values of S_{po} and P_{po}. Different choices for this point will only alter the position of the mapping in the ΔX, ΔY plane. The units on the ΔX, ΔY axes in the ΔX,ΔY plane must be the same as those on the X,Y axes in the mapping of the dominant closed loop pole region in the X,Y plane. When the mapping of the plant pole variation in the ΔX, ΔY plane is transferred to the X,Y plane, its angular position with respect to the ΔX, ΔY axes must be preserved, i.e., the mapping of the plant pole variation may not be rotated in the X,Y plane. Additional features of this mapping operation are covered in Section 9 of this chapter.

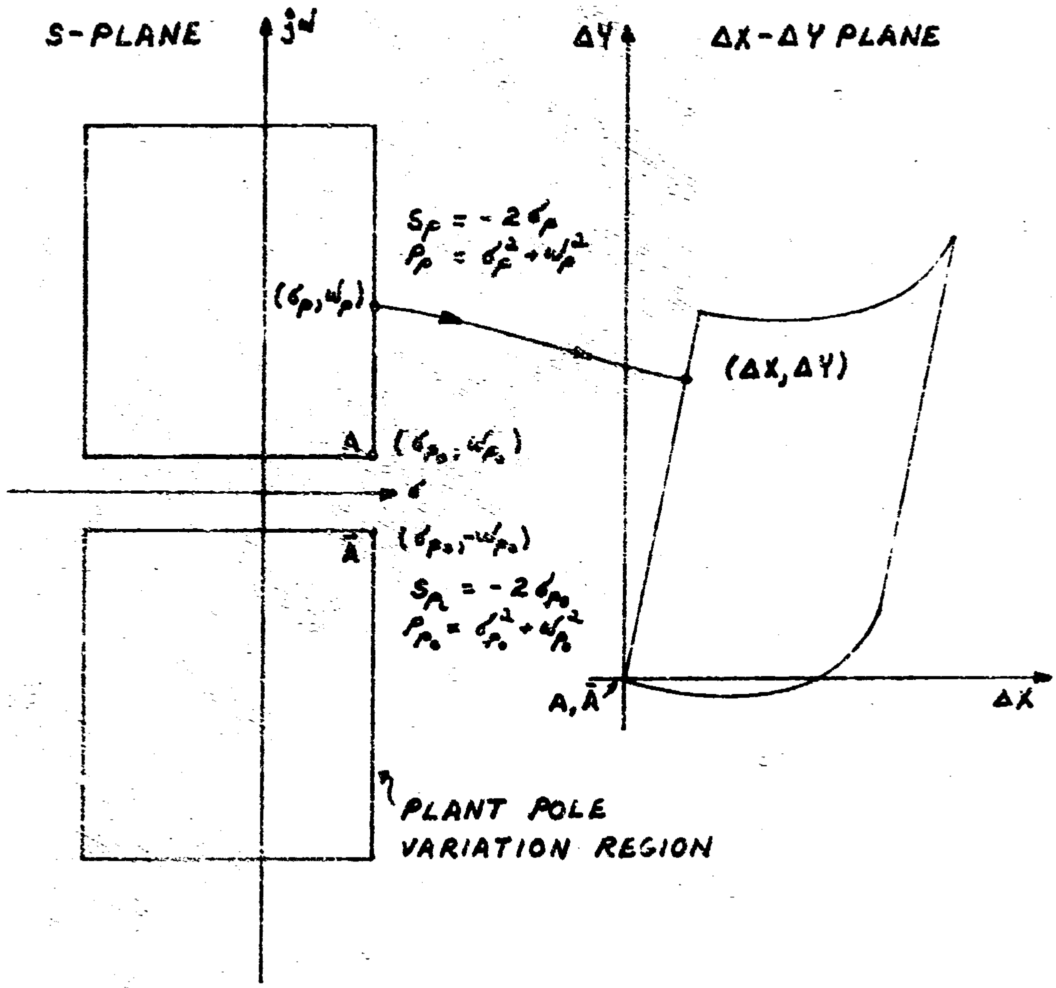
FIG. 3.3 MAPPING OF THE PLANT POLE VARIATION REGION IN THE S-PLANE INTO THE ΔX-ΔY PLANE

MAPPING EQUATIONS

$$\Delta X = P_1 (S_p - S_{p_0}) + P_p - P_{p_0}$$

$$\Delta Y = P_1 (P_p - P_{p_0})$$

PARAMETER P_1



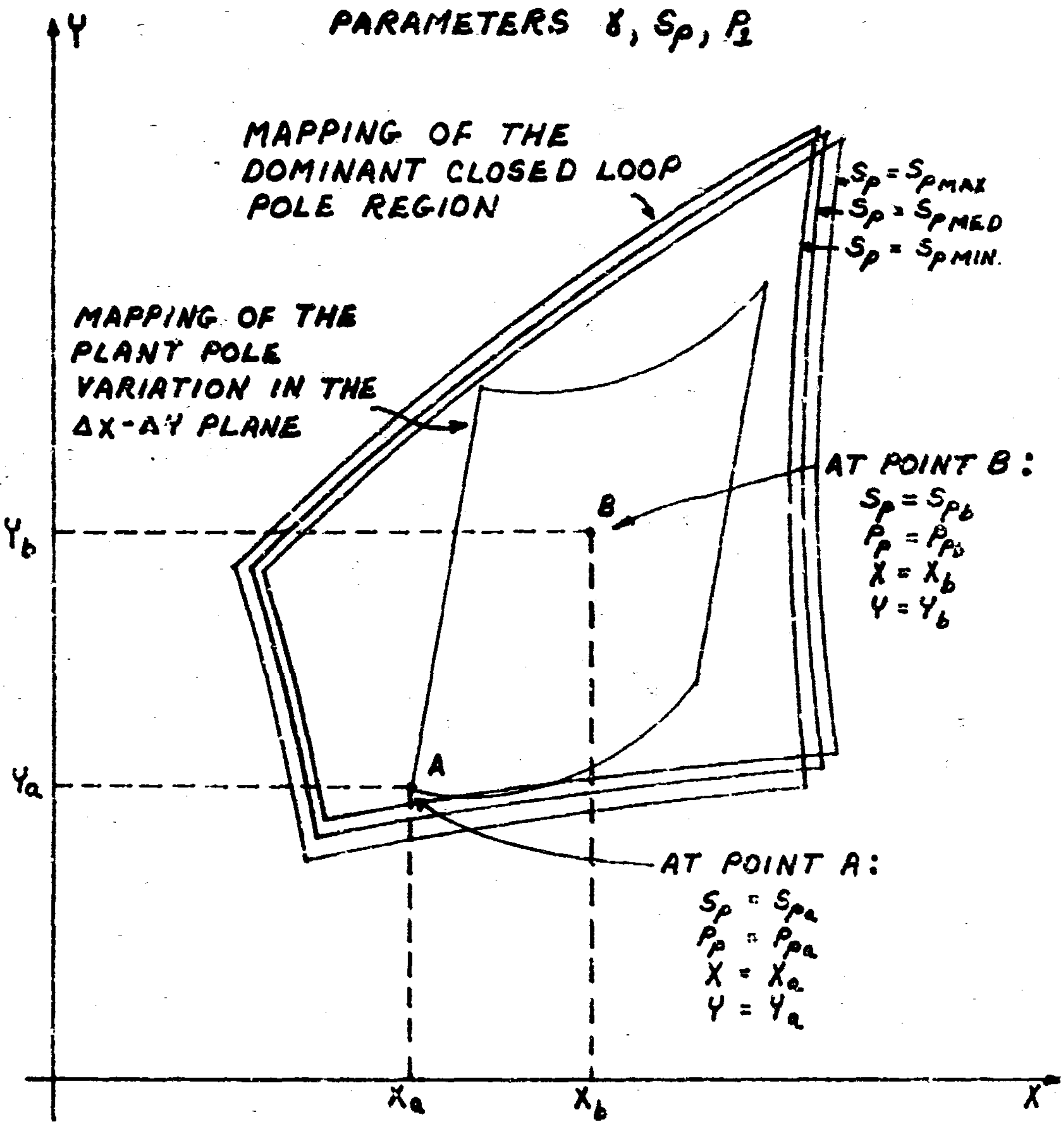
The problem now is to fit the mapping of the plant pole variation in the ΔX, ΔY plane into the interior of the mapping of the dominant closed loop pole region in the X,Y plane as shown in Fig. 3.4. If this fit is not possible, the mapping of the dominant closed loop pole region will have to be performed for larger values of γ.

3.6 Calculation of the System Gain and the Compensation Zero Location

Once the mapping of the plant pole variation fits inside the mapping of the dominant closed loop pole region in the X,Y plane, the value of the system gain kK and the compensation zero positions, given by the parameters S_o and P_o, may be computed. For a linear, time invariant, minimum-phase system, a value of system gain kK can always be found such that the mapping of the plant pole variation will fit inside the mapping of the dominant closed loop pole region in the X,Y plane.[7]

Figure 3.4 shows the mapping of the plant pole variation fitted inside the mapping of the dominant closed loop pole region in the X,Y plane. To solve for the values of kK, S_o and P_o, a point on the boundary of the mapping of the plant pole variation is chosen where the values of S_p and P_p are known. In Fig. 3.4, this point is denoted by A. The values of S_p and P_p at point A are, from Fig. 3.3 and Eqs. 3.27 and 3.28,

FIG. 3.4 CALCULATION OF THE SYSTEM GAIN AND THE ZERO LOCATION FROM THE MAPPINGS OF THE DOMINANT CLOSED LOOP POLE REGION AND THE PLANT POLE VARIATION



respectively, S_{p_a} and P_{p_a}. Denote the coordinates of point A in the X,Y plane as X_a and Y_a. Since γ and P_1 are known for this particular mapping, kh, S_o and P_o may be obtained from Eqs. 3.11, 3.12 and 3.13, as follows:

$$kh = X_a - S_{p_a} P_1 - P_{p_a} \tag{3.31}$$

$$P_o = \frac{\gamma}{kh} \tag{3.32}$$

$$S_o = \frac{Y_a - P_{p_a} P_1}{kh} \tag{3.33}$$

The value of kh should be interpreted as the necessary value of system gain, i.e. $k_{min} h$. The actual value of added gain to the system is, from Eq. 3.31

$$k = \frac{X_a - S_{p_a} P_1 - P_{p_a}}{k_{min}} \tag{3.34}$$

The choice of the point used to compute the values of kh, S_o and P_o has no effect on the values obtained for these quantities, so long as the point is on or within the mapping of the plant pole variation in the X,Y plane. To prove this, a second point B is chosen, as shown in Fig. 3.4. The value of system gain using point B is, from Eq. 3.31

$$kh = X_b - S_{p_b} P_1 - P_{p_b} \tag{3.35}$$

Now S_{p_b} and P_{p_b} are not known but they can be computed using Eqs. 3.29 and 3.30, i.e.

$$\Delta X = X_b - X_a = P_1(S_{p_b} - S_{p_a}) + (P_{p_b} - P_{p_a}) \tag{3.36}$$

$$\Delta Y = Y_b - Y_a = P_1(P_{p_b} - P_{p_a}) \tag{3.37}$$

Solving Eqs. 3.36 and 3.37 for S_{p_b} and P_{p_b} results in

$$P_{p_b} = \frac{Y_b - Y_a}{P_1} + P_{p_a} \tag{3.38}$$

$$S_{p_b} = \frac{X_b - X_a - (Y_b - Y_a)/P_1}{P_1} + S_{p_a} \tag{3.39}$$

Substitution of Eqs. 3.38 and 3.39 into Eq. 3.35 yields

$$k\lambda = X_b - X_b + X_a + \frac{Y_b - Y_a}{P_1} - S_{p_a} P_1 - \frac{Y_b - Y_a}{P_1} - P_{p_a}$$

$$= X_a - S_{p_a} P_1 - P_{p_a}$$

This equation is identical with equation 3.31, which implies that the value of $k\lambda$ is not dependent on the point used to compute it. From Eqs. 3.32 and 3.33, the same statement can be seen to hold for S_o and P_o.

3.7 Mathematical Explanation of the Design Procedure

This section presents the mathematical justification for the design procedure presented in Section 3.3. Referring to Fig. 3.5, the dominant closed loop pole regions in the s-plane are denoted by C and $\overline{C}$. The mapping function, which maps the dominant closed loop pole region into the X,Y plane, is denoted by T.

FIG. 3.5 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION, $T_i : C, \bar{C} \rightarrow S_i$ FOR $S_p = S_{p_i} \in [S_{PMIN}, S_{PMAX}]$

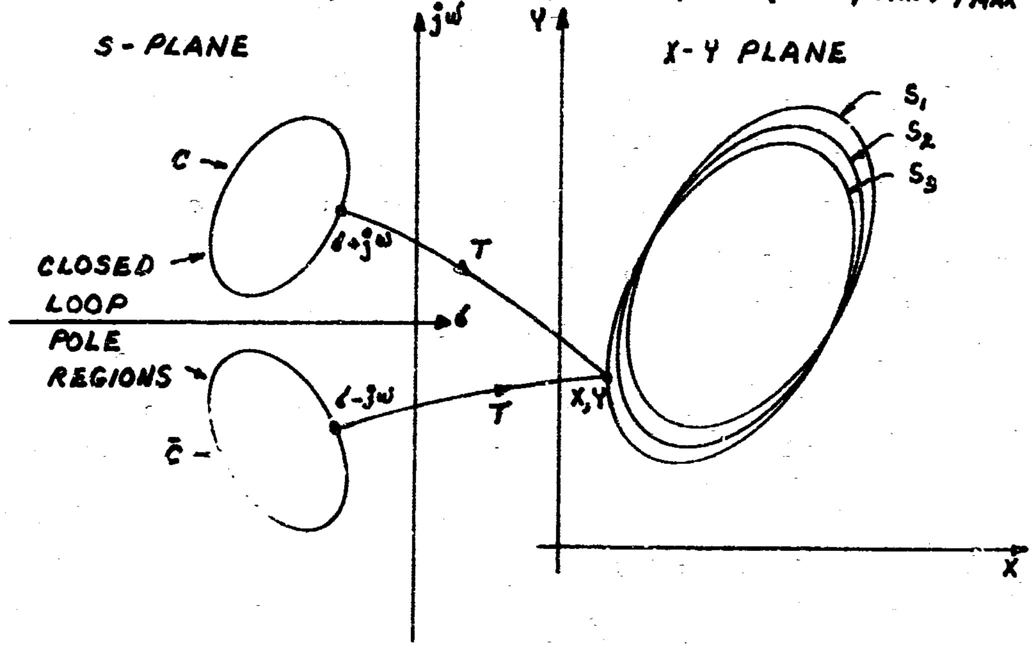
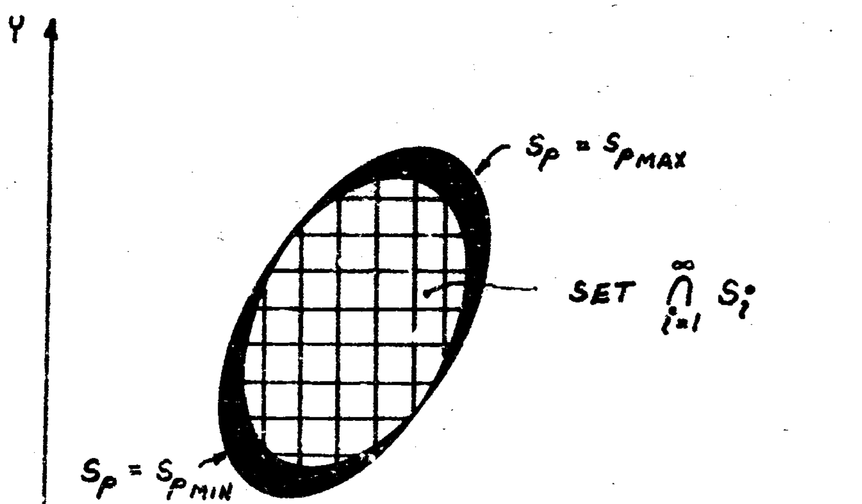


FIG. 3.6 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION FOR ALL $S_p \in [S_{PMIN}, S_{PMAX}]$



Denote the mapping of the sets C, $\bar{C}$ for $S_p = S_{p_i} \epsilon [S_{pmin}, S_{pmax}]$ by means of Eqs. 3.14 and 3.15, as S_i i.e. T_i: $C, \bar{C} \rightarrow S_i$ (See Fig. 3.5). Now from Eqs. 3.14 and 3.15, it is evident that no two pair of points in $C, \bar{C}$ map into the same point in S_i, therefore T_i is a <u>one-to-one</u> mapping function. Since every element of S_i appears as the image of at least one pair of points in $C, \bar{C}$, T_i maps C and $\bar{C}$ <u>onto</u> S_i. Now, since T_i is a <u>one-to-one</u> mapping function and also maps C and $\bar{C}$ <u>onto</u> S_i, then the inverse mapping function T_i^{-1} exists and maps S_i <u>onto</u> $C, \bar{C}$ in a <u>one-to-one</u> fashion.

The mapping of C and $\bar{C}$ into the X,Y plane for an infinite number of values of S_p between S_{pmin} and S_{pmax} will result in the set shown in Fig. 3.6. This set may be represented as

$$S = \bigcap_{i=1}^{\infty} S_i \tag{3.36}$$

Under the assertion that T_i^{-1} exists and maps S_i <u>onto</u> $C, \bar{C}$ in a <u>one-to-one</u> fashion, i.e.,

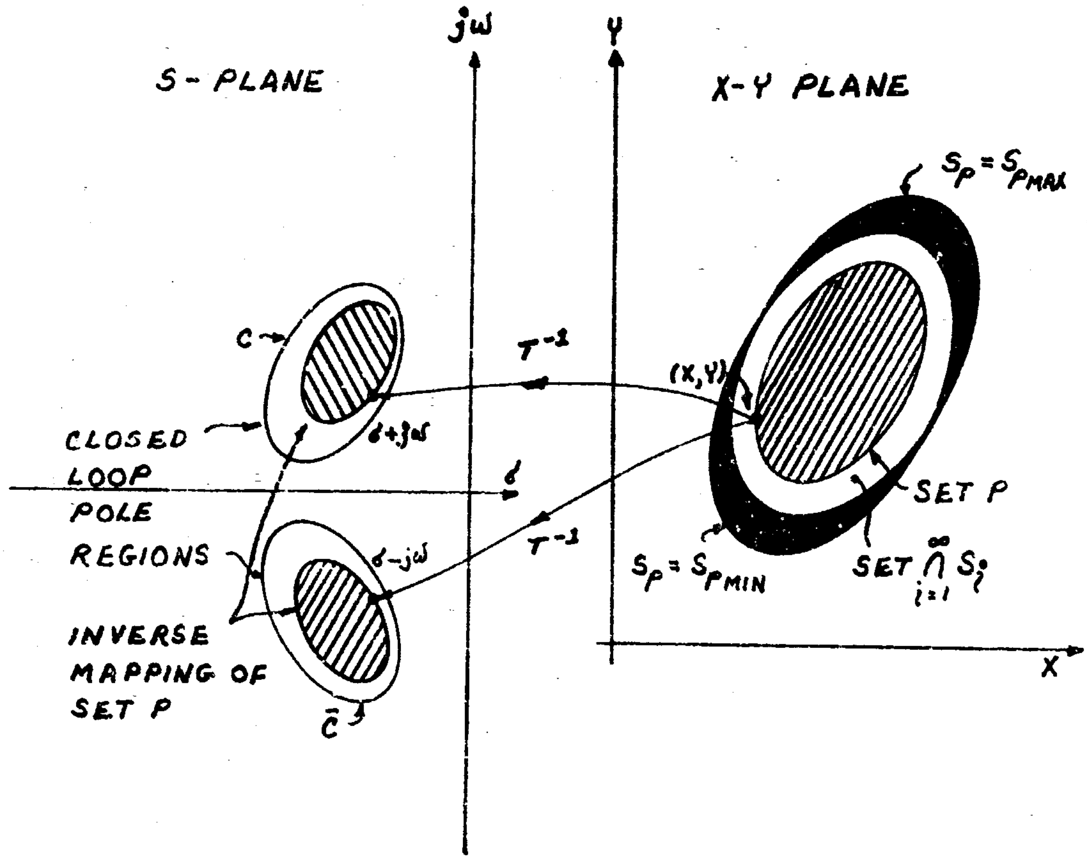
$$T_i^{-1} : S_i \rightarrow C, \bar{C} \quad \text{for } S_p = S_{p_i} \epsilon [S_{pmin}, S_{pmax}] \tag{3.37}$$

it follows that

$$T_i^{-1} : \bigcap_{i=1}^{\infty} S_i \rightarrow C, \bar{C} \quad \text{for all } S_{p_i} \epsilon [S_{pmin}, S_{pmax}] \tag{3.38}$$

Referring to Fig. 3.7, this means that if the given variation in X and Y, denoted by the set P, is a

FIG. 3.7 INVERSE MAPPING OF THE SET P IN THE X-Y PLANE TO THE S-PLANE



subset of the set $S = \bigcap_{i=1}^{\infty} S_i$, then the inverse mapping of P into the s-plane will be contained in the sets $C, \overline{C}$. Letting the set P represent the mapping of the plant pole variation, then the solution of Eqs. 3.31, 3.32 and 3.33 yield the necessary values of kh, S_o and P_o to position the set P within the set S which represents the mapping of the dominant closed loop pole region. In this manner the difficult problem of choosing kh, S_o and P_o so as to insure that the dominant closed loop poles lie within their acceptable region in the s-plane, is transformed into the less difficult problem of determining kh, S_o and P_o such that the mapping of the plant pole variation in the ΔX, ΔY plane may be fitted inside the mapping of the acceptable dominant pole region in the X,Y plane.

3.8 Analytic Aspects of the Mapping of the Dominant Closed Loop Pole Region

The relative complexity of Eqs. 3.14 and 3.15 require that the actual mapping of the dominant closed loop pole region be done point by point using a digital computer. In this section, the mapping of curves of constant P_r and curves of constant S_r in the s-plane into the X,Y plane is examined. Curves of constant P_r in the s-plane are shown in Fig. 3.8. The mapping equations are

FIG. 3.8 CURVES OF CONSTANT P_r IN THE S-PLANE

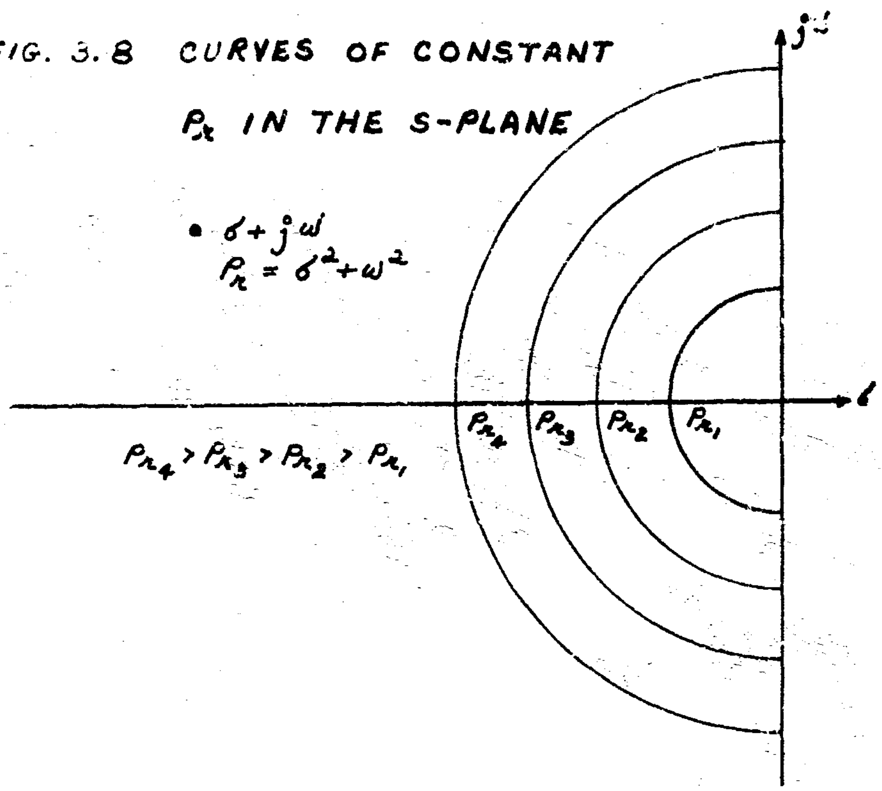
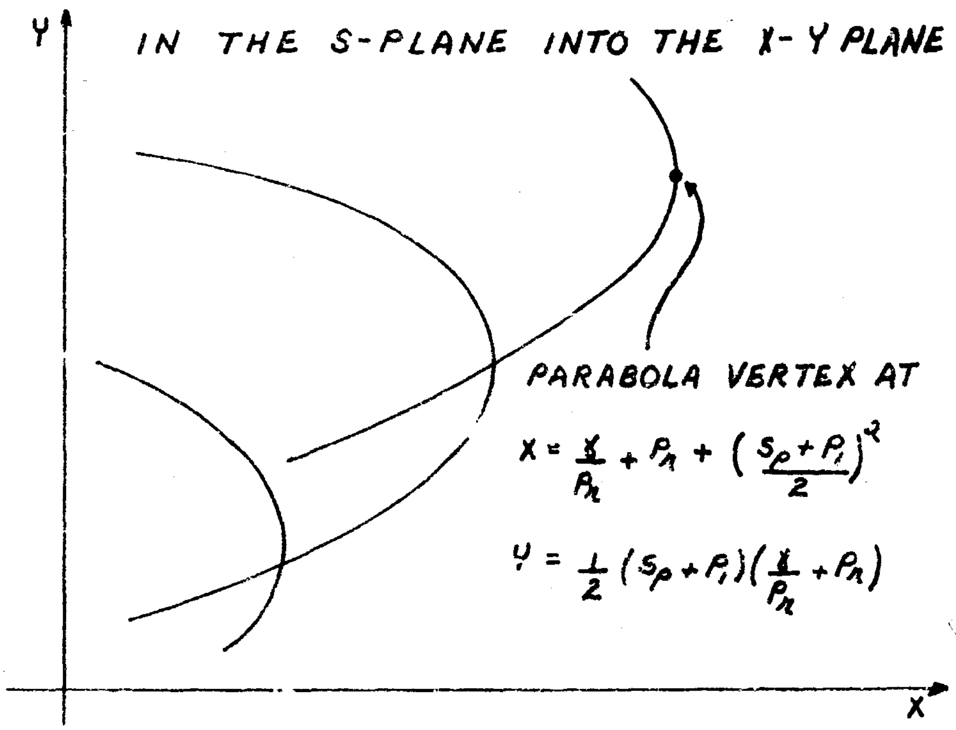


FIG. 3.9 MAPPING OF CURVES OF CONSTANT P_r IN THE S-PLANE INTO THE X-Y PLANE



$$\frac{\gamma}{P_r} + S_r(S_p+P_1-S_r)+P_r = X$$

$$\frac{S_r\gamma}{P_r} + P_r(S_p+P_1-S_r) = Y$$

Since P_r is to be held constant, S_r must be eliminated in these two equations. The parameters γ, P_1 and S_p will also be held constant in these two equations. Solving the second equation for S_r yields

$$S_r = \frac{Y-P_rP_1-P_rS_p}{\gamma/P_r - P_r} \qquad (3.39)$$

Substitution of Eq. 3.39 into the first equation gives

$$\frac{\gamma}{P_r}+\{\frac{Y-P_rP_1-P_rS_p}{\gamma/P_r - P_r}\}(S_p+P_1)-\{\frac{Y-P_rP_1-P_rS_p}{\gamma/P_r - P_r}\}^2+P_r = X \qquad (3.40)$$

After considerable algebraic manipulation, Eq. 3.40 may be placed in the form of

$$-(\gamma/P_r-P_r)^2\{X-\gamma/P_r-P_r-\frac{(S_p+P_1)^2}{4}\}$$

$$= \{Y-\tfrac{1}{2}(S_p+P_1)(\gamma/P_r+P_r)\}^2 \qquad (3.41)$$

Equation 3.41 is in the form

$$-4a(X-h) = (Y-k)^2 \qquad (3.42)$$

which is the equation of a parabola opening to the left, whose vertex is at (h,k) with a focal length equal to a.[8] Relating these quantities to Eq. 3.41, the vertex

of the parabola, represented by Eq. 3.41, has the following coordinates in the X,Y plane

$$X = \gamma/P_r + P_r + \frac{(S_p+P_1)^2}{4} \qquad (3.43)$$

$$Y = \tfrac{1}{2}(S_p+P_1)(\gamma/P_r+P_r) \qquad (3.44)$$

The focal length of this parabola is

$$a = \frac{(\gamma/P_r-P_r)^2}{4} \qquad (3.45)$$

The position of a parabola defined by Eq. 3.41 is shown in the X,Y plane in Fig. 3.9. It should be noted that Eq. 3.41 represents a parabola whose axis is parallel with the X axis in the X,Y plane. This is indicated by the absence of any terms of the form CXY in Eq. 3.41 with C = constant.

Curves of constant S_r in the s-plane are shown in Fig. 3.10. Since S_r is to be held constant, P_r must be eliminated in the two mapping equations. Defining

$$A \triangleq S_p+P_1-S_r \qquad (3.46)$$

and solving the second mapping equation for P_r yields

$$P_r = \frac{Y}{2A} \pm \sqrt{\frac{Y^2}{4A^2} - \frac{S_rY}{A}} \qquad (3.47)$$

Substitution of Eq. 3.47 into the first mapping equation gives

FIG. 3.10 CURVES OF CONSTANT S_R IN THE S-PLANE

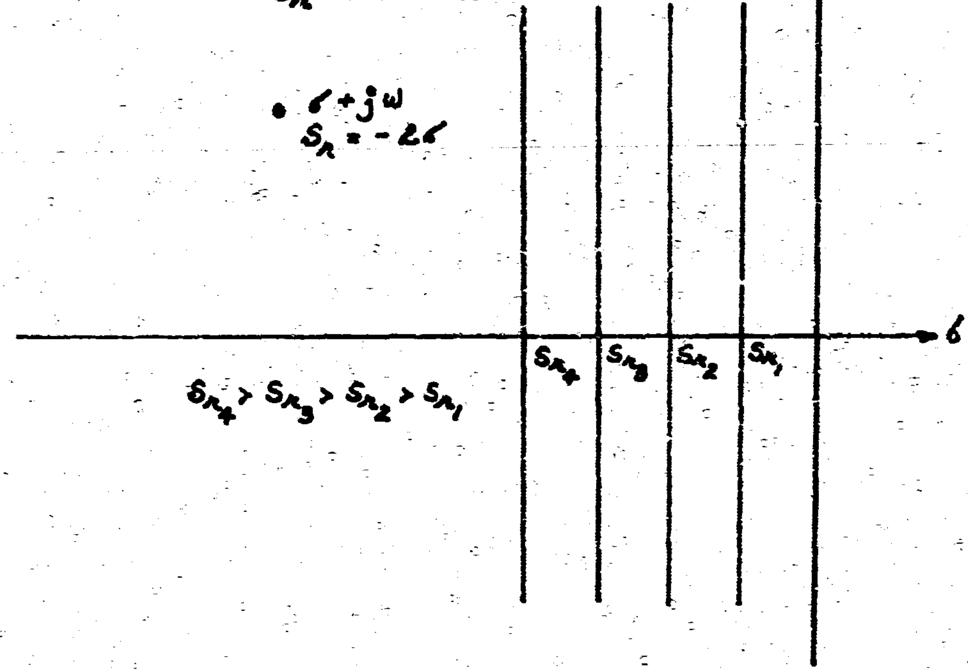
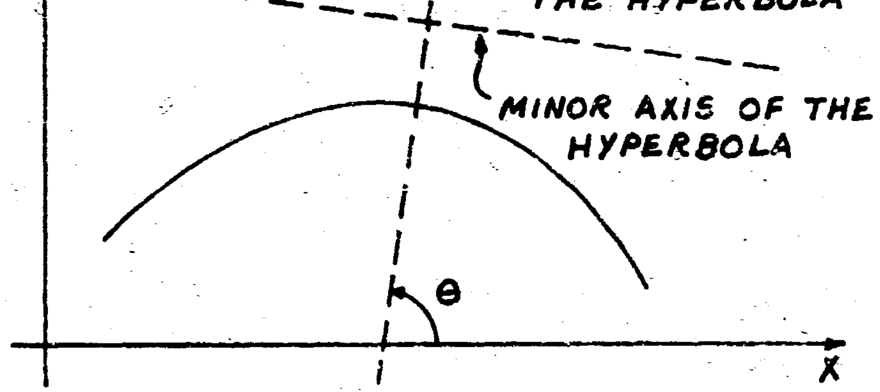


FIG. 3.11 MAPPING OF CURVES OF CONSTANT S_R IN THE S-PLANE INTO THE X-Y PLANE



$$\frac{Y}{\frac{Y}{2A} \pm \sqrt{\frac{Y^2}{4A^2} - \frac{S_r Y}{A}}} + AS_r + \frac{Y}{2A} \pm \sqrt{\frac{Y^2}{4A^2} - \frac{S_r Y}{A}} = X \tag{3.48}$$

After much algebraic manipulation, this equation may be put in the following form

$$X^2 - XY\left(\frac{S_p+P_1}{S_r A}\right) + \frac{1}{4}\frac{Y^2}{S_r A} - 2AS_r X + Y(S_p+P_1) + A^2 S_r^2 + \frac{Y(A-S_r)^2}{S_r A} = 0 \tag{3.49}$$

This equation now is in the form of

$$aX^2 + 2bXY + cY^2 + 2dX + 2eY + f = 0 \tag{3.50}$$

with

$$a = 1$$

$$b = -\frac{1}{2}\left(\frac{S_p+P_1}{S_r A}\right)$$

$$c = \frac{1}{4}\frac{1}{S_r A}$$

which is of the form of a second degree equation in a rotated system of coordinates.[8] The type of second degree equation which Eq. 3.50 represents can be determined by examining the coefficients a, b and c. From Eq. 3.50

$$b^2 - ac = \left\{-\frac{1}{2}\left(\frac{S_p+P_1}{S_r A}\right)\right\}^2 - \frac{1}{4}\frac{1}{S_r A}$$

$$= 4\, S_r^2 A^2 \{(S_p+P_1)^2 - S_r A\} \tag{3.51}$$

It can be shown[8] that the following relationships exist between the coefficients of Eq. 3.50 and the type of second degree equation.

$$b^2 - ac < 0 \rightarrow \text{an ellipse}$$

$$b^2 - ac = 0 \rightarrow \text{a parabola}$$

$$b^2 - ac > 0 \rightarrow \text{a hyperbola}$$

Equation 3.51 can be put in the form

$$4S_r^2A^2\{(S_r+A)^2-S_rA\}$$

or
$$4S_r^2A^2\{S_r^2+S_rA+A^2\} \tag{3.52}$$

Now if A is greater than zero, so is Eq. 3.52, i.e.,

$$4S_r^2A^2(S_r^2+S_rA+A^2) > 0 \tag{3.53}$$

since S_r is always greater than zero (closed loop poles in left hand plane) For A to be greater than zero, S_p, P_1 and S_r must satisfy the following

$$P_1 > S_r-S_p \tag{3.54}$$

The parameter S_p is negative if the plant poles lie in the right half plane. Therefore Eq. 3.54 can be written as

$$P_1>S_r+|S_p| \quad \text{for } S_p<0 \tag{3.55}$$

This equation is certainly satisfied for most feedback control systems. Curves of constant S_r in the s-plane therefore map as hyperbolas in the X,Y plane under the

mapping equations 3.14 and 3.15. The position of a hyperbola defined by Eq. 3.49 in the X,Y plane is shown in Fig. 3.11.

The angle which the major axis of the hyperbola is rotated from the X axis in the X,Y plane is given by[8]

$$\theta = \mathrm{Tan}^{-1}\left\{\frac{(c-a) \pm \sqrt{(c-a)^2+4b^2}}{2b}\right\} \tag{3.56}$$

$$= \mathrm{Tan}^{-1}\left\{\frac{(1/4S_rA-1) \pm\sqrt{(1/4S_rA-1)^2+\{(A+S_r)/(S_rA)\}^2}}{-\left(\frac{A+S_r}{S_rA}\right)}\right\} \tag{3.57}$$

To obtain an insight on the magnitude of the angle θ, the following values are assigned to the parameters in Eq. 3.57 which are typical for the design example in Chapter II and the design example that will be considered in this chapter

$$S_r = 10 \qquad S_p = 2$$

$$P_1 = 30 \qquad A = 22$$

Substitution of these quantities into Eq. 3.57 gives

$$\theta = \mathrm{Tan}^{-1}\left\{\frac{(1/880-1) \pm \sqrt{(1/880-1)^2+(32/220)^2}}{-\left(\frac{32}{220}\right)}\right\}$$

$$= \mathrm{Tan}^{-1}\ 13.81 = 85.86°$$

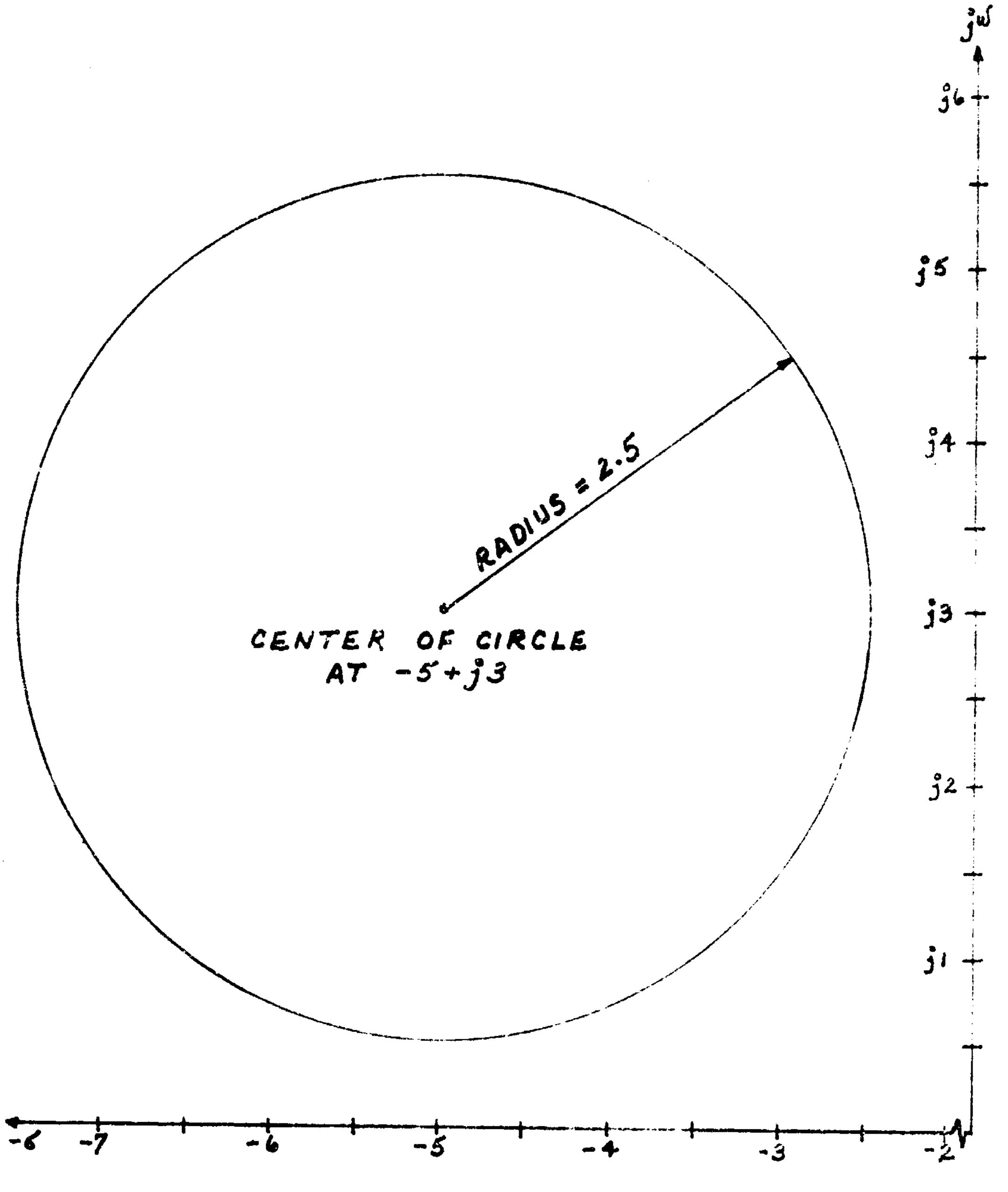
For the range of parameters considered in this paper, a good approximation to Eq. 3.57 is

$$\theta \cong \mathrm{Tan}^{-1} \frac{2S_r A}{S_r + A} \qquad (3.58)$$

The sign in Eq. 3.57 may be arbitrarily chosen to make θ positive.

The only other two parameters in the mapping of the dominant closed loop pole region are P_1, the "far-off" pole on the real axis, and γ defined by Eq. 3.11. The qualitative effects of these parameters on the mapping of the dominant closed loop pole region are shown in Figs. 3.13 and 3.14. The dominant closed loop pole region used for this investigation is shown in Fig. 3.12. From inspection of Figs. 3.13 and 3.14, it is seen that increasing the value of γ both increases the relative size of the mapping of the dominant closed loop pole region and its coordinates in the X,Y plane. Increasing the value of P_1 merely increases the coordinates of the mapping in the X,Y plane. Figure 3.15 illustrates the variation in the "far-off" closed loop poles, p_{f_1} and p_{f_2}, during the mapping of dominant closed loop region into the X,Y plane. These "far-off" poles are obtained using Eqs. 3.25 and 3.26 and the procedure outlined in Section 4 during the mapping operation.

FIG. 3.12 DOMINANT CLOSED LOOP POLE REGION USED TO ILLUSTRATE THE EFFECT OF γ AND P_1 ON THE MAPPING INTO THE X-Y PLANE



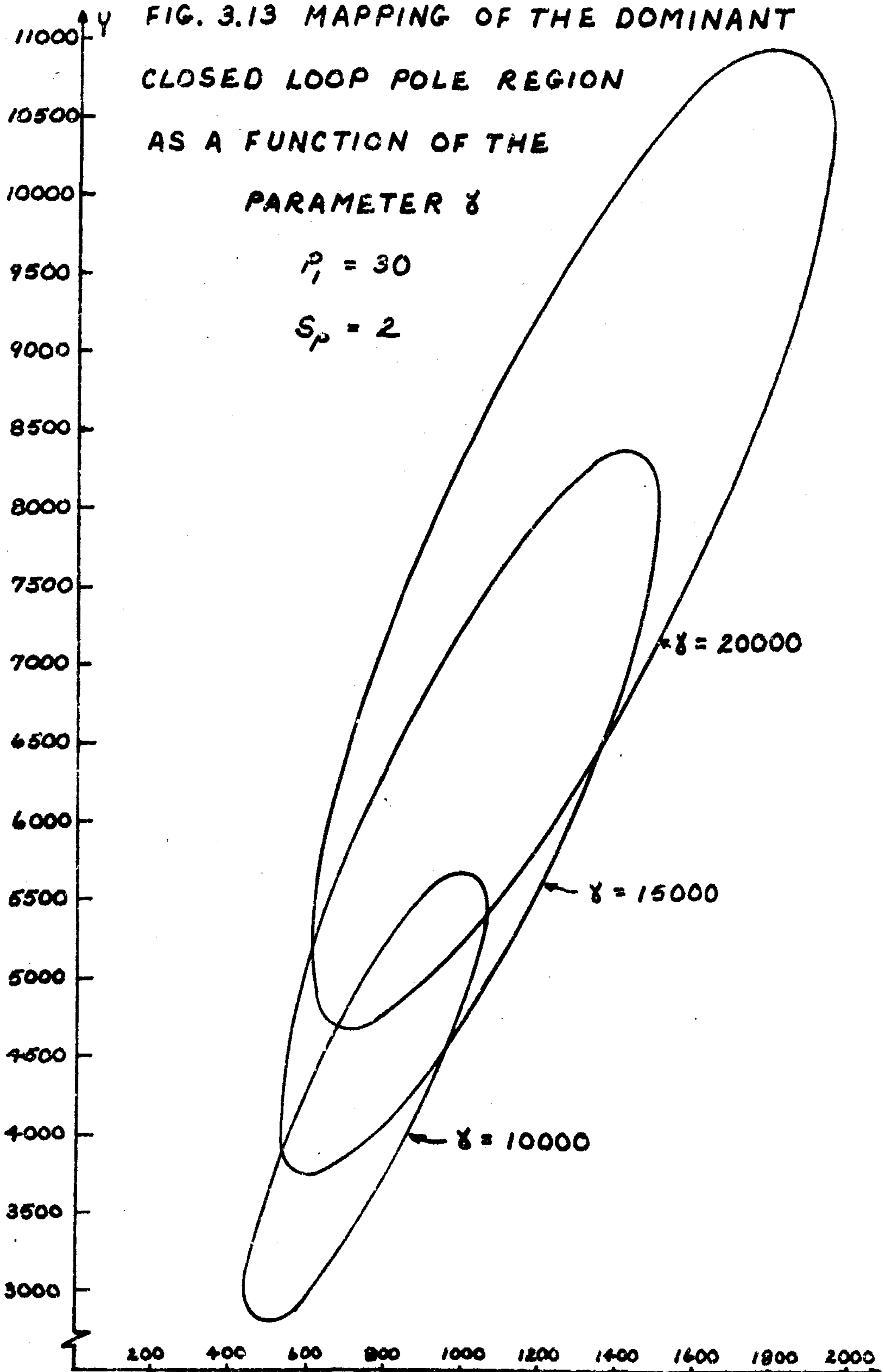


FIG. 3.13 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION AS A FUNCTION OF THE PARAMETER γ

$P_1 = 30$

$S_p = 2$

FIG. 3.14 MAPPING OF THE CLOSED LOOP POLE REGION AS A FUNCTION OF THE OPEN LOOP POLE P_1

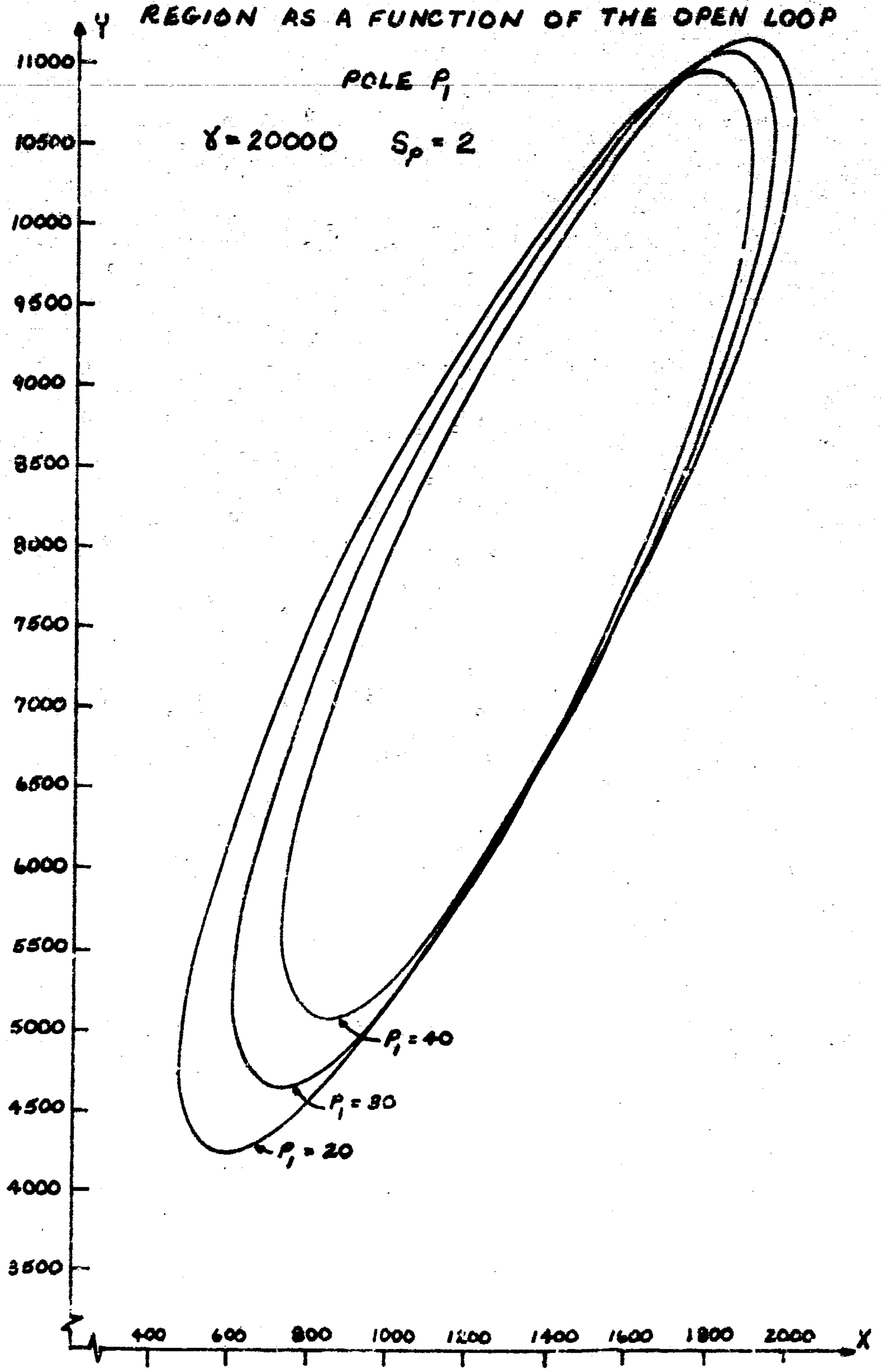
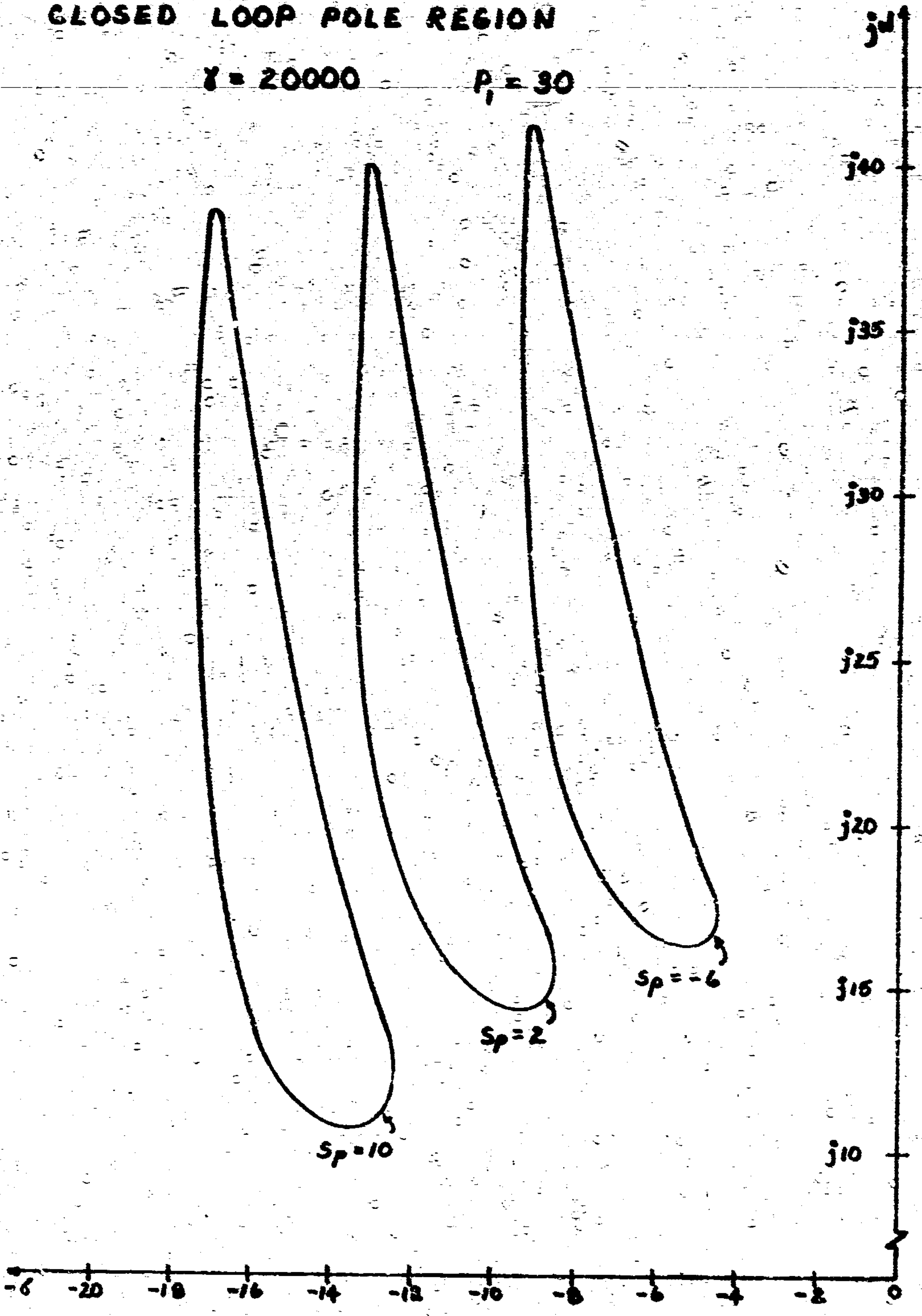


FIG. 3.15 VARIATION IN THE FAR-OFF CLOSED LOOP POLE P_{f_1} FOR THE MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION

$\gamma = 20000 \qquad P_1 = 30$



3.9 Analytic Aspects of the Mapping of the Plant Pole Variation

As with mapping of the dominant closed loop pole region, the mapping of the plant pole variation (Eqs. 3.29, 3.30) is most easily accomplished by mapping point by point using a digital computer. In this section, the mapping of curves of constant P_p, constant S_p and constant ω in the s-plane into the ΔX, ΔY plane is investigated.

The mapping equations are

$$\Delta X = P_1 \Delta S_p + \Delta P_p = P_1(S_p - S_{p_o}) + (P_p - P_{p_o})$$

$$\Delta Y = P_1 \Delta P_p = P_1(P_p - P_{p_o})$$

where S_{p_o} and P_{p_o} define an arbitrary point $\sigma_{p_o} + j\omega_{p_o}$ on the boundary of the plant pole variation in the s-plane.

Curves of constant P_p in the s-plane are shown in Fig. 3.16. The mapping equations with P_p equal to a constant are

$$\Delta X = P_1(S_p - S_{p_o}) + P_p - P_{p_o} \qquad (3.58)$$

$$\Delta Y = P_1(P_p - P_{p_o}) = \text{constant} \qquad (3.59)$$

The mapping of curves of constant P_p in the s-plane into the X,Y plane is shown in Fig. 3.17.

FIG. 3.16 CURVES OF CONSTANT P_p IN THE S-PLANE

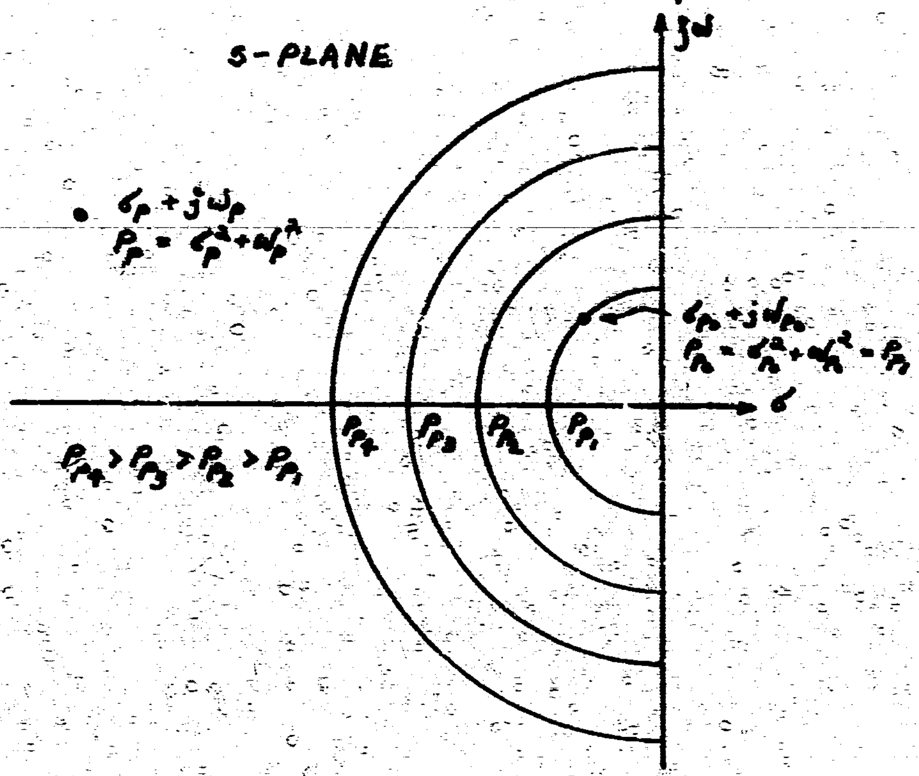
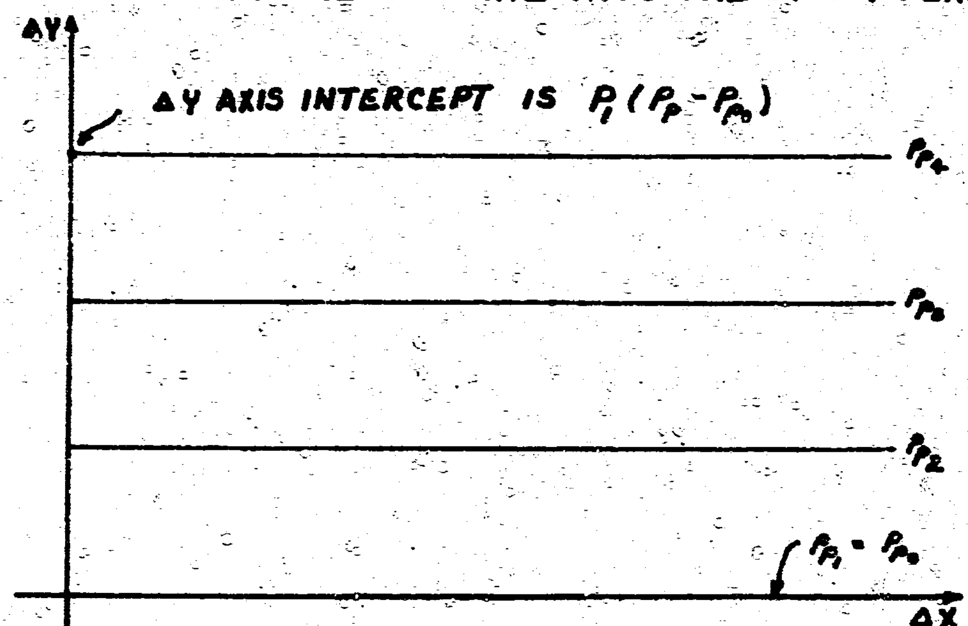


FIG. 3.17 MAPPING OF CURVES OF CONSTANT P_p IN THE S-PLANE INTO THE ΔX-ΔY PLANE



Curves of constant S_p in the s-plane are shown in Fig. 3.18. Eliminating P_p in the two mapping equations yields

$$\Delta Y = P_1 \Delta X - P_1^2 (S_p - S_{p_o}) \qquad (3.60)$$

Equation 3.60 is in the form

$$\Delta Y = m\, \Delta X + b \qquad (3.61)$$

which is the equation of a straight line in the $\Delta X, \Delta Y$ plane with slope m and ΔY intercept b where

$$m = P_1 \quad ; \quad b = -P_1^2 (S_p - S_{p_o})$$

The mapping of curves of constant S_p in the s-plane into the $\Delta X, \Delta Y$ plane is shown in Fig. 3.19.

Curves of constant ω are shown in Fig. 3.20. A point on the $\omega = \omega_1$ line is defined as $\sigma + j\omega_1$. The mapping equations are

$$\begin{aligned} \Delta X &= P_1 \Delta S_p + \Delta P_p \\ &= P_1(-2\sigma + 2\sigma_{p_o}) + (\sigma^2 + \omega_1^2 - \sigma_{p_o}^2 - \omega_{p_o}^2) \end{aligned} \qquad (3.62)$$

$$\begin{aligned} \Delta Y &= P_1 \Delta P_p \\ &= P_1(\sigma^2 + \omega_1^2 - \sigma_{p_o}^2 - \omega_{p_o}^2) \end{aligned} \qquad (3.63)$$

Since ω_1 is to be held constant, σ must be eliminated in Eqs. 3.62 and 3.63. Solving equation 3.63 for σ yields

$$\sigma = \pm \sqrt{\frac{\Delta Y}{P_1} + \sigma_{p_o}^2 + \omega_{p_o}^2 - \omega_1^2} \qquad (3.64)$$

Substitution of Eq. 3.64 into Eq. 3.62 gives

FIG. 3.18 CURVES OF CONSTANT S_p IN THE S-PLANE

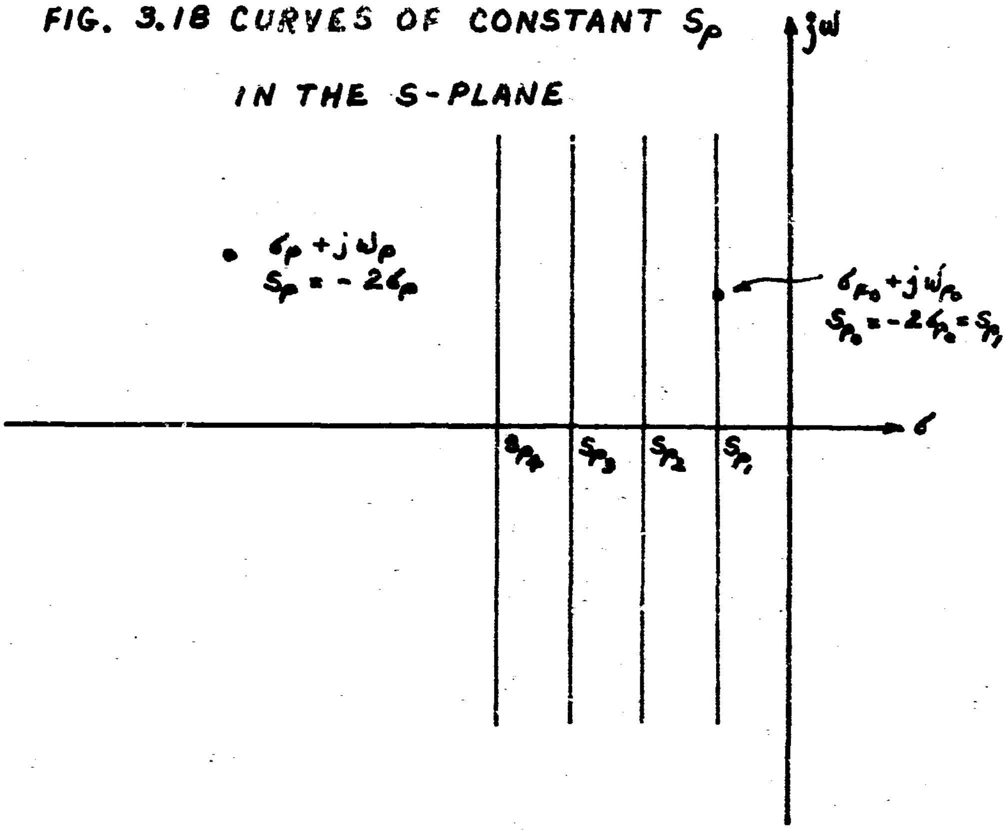


FIG. 3.19 MAPPING OF CURVES OF CONSTANT S_p IN THE S-PLANE INTO THE ΔX,ΔY PLANE

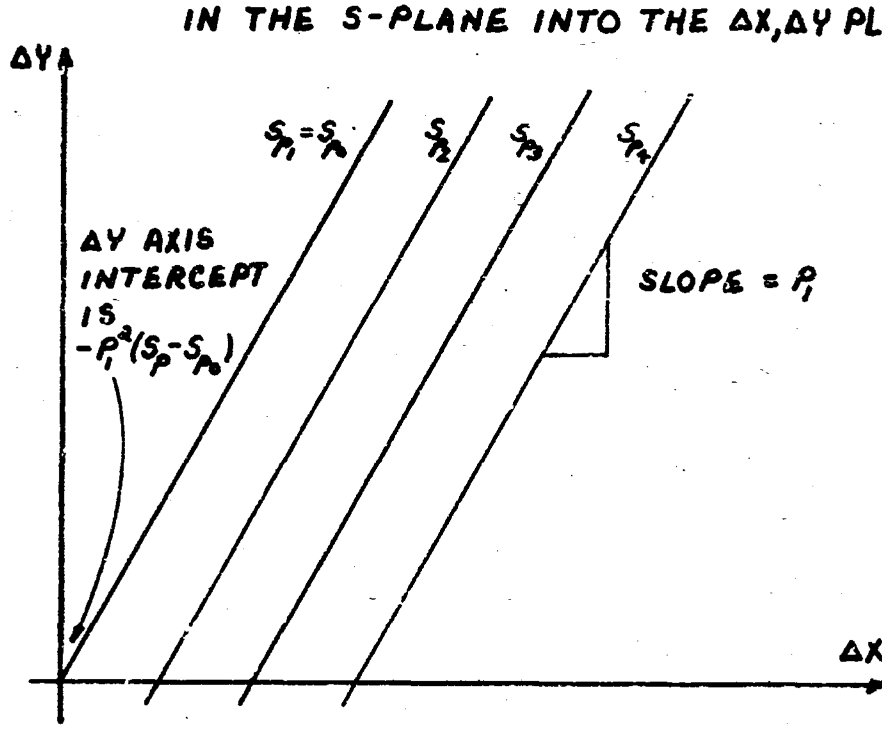


FIG. 3.20 CURVES OF CONSTANT ω IN THE S-PLANE

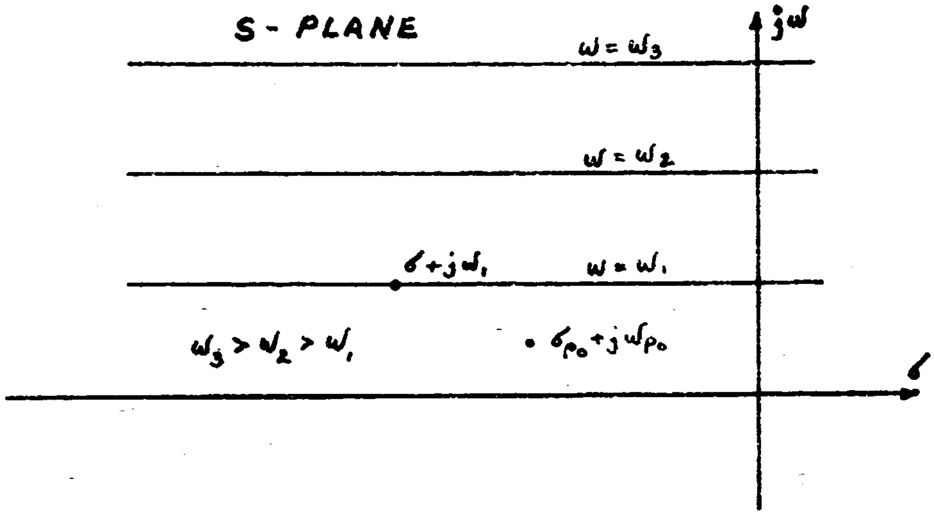
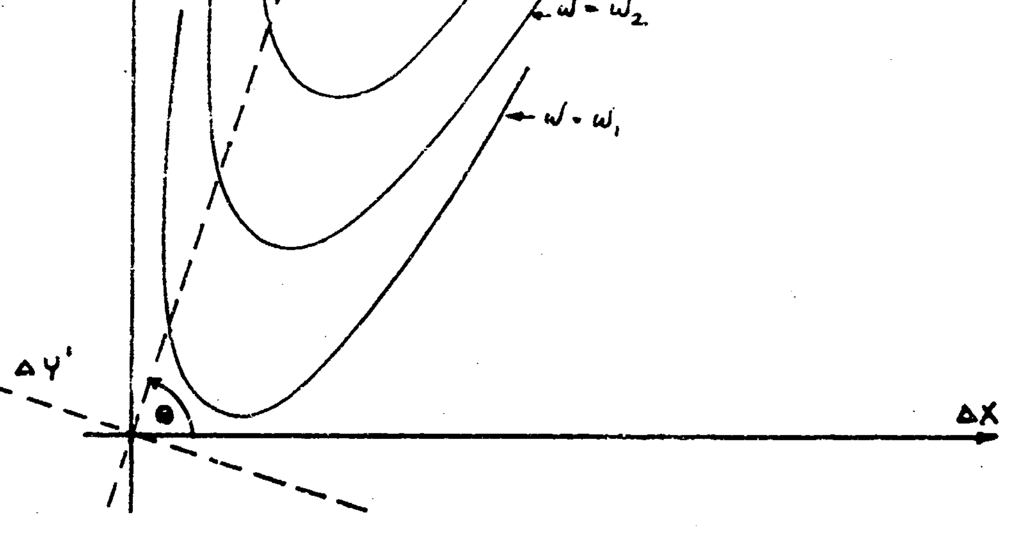


FIG. 3.21 MAPPING OF CURVES OF CONSTANT ω IN THE S-PLANE INTO THE ΔX-ΔY PLANE



$$\Delta X = -2P_1\{\pm\sqrt{\frac{\Delta Y}{P_1}+\sigma^2_{p_o}+\omega^2_{p_o}-\omega^2_1}-\sigma_{p_o}\}+\frac{\Delta Y}{P_1} \qquad (3.65)$$

Rearranging Eq. 3.65 gives

$$(\Delta X)^2-\frac{2\Delta X\Delta Y}{P_1}+\frac{(\Delta Y)^2}{P_1^2}-4P_1\sigma_{p_o}\Delta X+4\Delta Y(\sigma_{p_o}-P_1)-4P_1^2(\omega^2_{p_o}-\omega^2_1)=0 \qquad (3.66)$$

This equation is now in the form of Eq. 3.50, the form of a second degree equation in a rotated system of coordinates. The form of this second degree equation may be determined by examining the first three coefficients, a, b, and c of Eq. 3.66. For Eq. 3.66

$$a = 1 \;;\quad b = -1/P_1 \;;\quad c = 1/P_1^2$$

and $$b^2 - ac = 1/P_1^2 - 1/P_1^2 = 0$$

According to the conditions previously stated, equation 3.66 represents a parabola in a rotated system of coordinates.

The angle which the axis of the parabola makes with the ΔX axis in the $\Delta X,\Delta Y$ plane is given by Eq. 3.56 which is

$$\theta = \text{Tan}^{-1}\{\frac{(c-a)\pm\sqrt{(c-a)^2+4b^2}}{2b}\}$$

$$= \text{Tan}^{-1}\{\frac{(1/P_1^2-1)\pm\sqrt{(1/P_1^2-1)^2+4/P_1^2}}{-2/P_1}\}$$

$$= \text{Tan}^{-1}P_1 \qquad (3.67)$$

where the sign on the radical has been chosen to make θ positive.

Unlike the mapping equations developed for the dominant closed loop pole region, equation 3.66 is not overly complex and a transformation may be made that will eliminate the $\Delta X \Delta Y$ term in Eq. 3.66. The transformation equations which will transform Eq. 3.66 into the following form

$$a'(\Delta X')^2+c'(\Delta Y')^2+2d'\Delta X'+2e'\Delta Y'+f = 0 \qquad (3.68)$$

where the primes indicate quantities referred to the rotated system of coordinates are[8]

$$a' = a\cos^2\theta+2b\sin\theta\cos\theta+c\sin^2\theta \qquad (3.69)$$

$$c' = a\sin^2\theta-2b\sin\theta\cos\theta+c\sin^2\theta \qquad (3.70)$$

$$d' = d\cos\theta+e\sin\theta \qquad (3.71)$$

$$e' = -d\sin\theta+e\cos\theta \qquad (3.72)$$

The trigonmetric functions may be expressed as follows

$$\cos\theta = \frac{1}{\sqrt{1+\mathrm{Tan}^2\theta}} = \frac{1}{\sqrt{1+P_1^2}} \qquad (3.73)$$

$$\sin\theta = \frac{\mathrm{Tan}\theta}{\sqrt{1+\mathrm{Tan}^2\theta}} = \frac{1}{\sqrt{1+P_1^2}} \qquad (3.74)$$

The new coefficients a', c', d' and e' are

$$a' = \frac{1}{1+P_1^2} - \frac{(2/P_1)P_1}{1+P_1^2} + \frac{(1/P_1^2)P_1^2}{1+P_1^2} = 0 \qquad (3.75)$$

$$c' = \frac{P_1^2}{1+P_1^2} + \frac{(2/P_1)P_1}{1+P_1^2} + \frac{1/P_1^2}{1+P_1^2} = \frac{P_1^2+1}{P_1^2} \qquad (3.76)$$

$$d' = \frac{-2P_1\sigma_{p_o}}{\sqrt{1+P_1^2}} + \frac{2P_1(\sigma_{p_o}-P_1)}{\sqrt{1+P_1^2}} = \frac{-2P_1^2}{\sqrt{1+P_1^2}} \qquad (3.77)$$

$$e' = \frac{2P_1^2\sigma_{p_o}}{\sqrt{1+P_1^2}} + \frac{2(\sigma_{p_o}-P_1)}{\sqrt{1+P_1^2}} = \frac{2\sigma_{p_o}(P_1^2+1)-2P_1}{\sqrt{1+P_1^2}} \qquad (3.78)$$

Equation 3.68 with $a'=0$ is of the following form

$$c'(\Delta Y')^2+2d'\Delta X'+2e'\Delta Y'+f = 0 \qquad (3.79)$$

which may be put in the form of

$$\left(\Delta Y'+\frac{e'}{c'}\right)^2 = -\frac{2d'}{c'}\left(\Delta X'+\frac{f}{2d'} - \frac{(e')^2}{2d'c'}\right) \qquad (3.80)$$

From Eq. 3.42, this equation represents a parabola in $\Delta X', \Delta Y'$ system of coordinates with its center at

$$\Delta X = -\frac{f}{2d'} + \frac{(e')^2}{2d'c'} \qquad (3.81)$$

$$\Delta Y' = -e'/c' \qquad (3.82)$$

and focal length a of

$$a = -\frac{d'}{2c'} \qquad (3.83)$$

In most feedback control systems, P_1 is much greater than unity. Therefore the following approximations can be used for the coefficients c', d' and e':

$$c' = \frac{P_1^2+1}{P_1^2} \cong 1 \qquad (3.84)$$

$$d' = -\frac{2P_1^2}{\sqrt{1+P_1^2}} \cong -2P_1 \qquad (3.85)$$

$$e' = \frac{2\sigma_{p_o}(1+P_1^2)-2P_1}{\sqrt{1+P_1^2}} \cong 2(\sigma_{p_o}P_1-1) \qquad (3.86)$$

The approximation to Eq. 3.80 is then

$$\{\Delta Y'+2(\sigma_{p_o}P_1-1)\}^2 = 4P_1\{\Delta X'+P_1(\omega_{p_o}^2-\omega_1^2)+\frac{(\sigma_{p_o}P_1-1)^2}{P_1}\} \qquad (3.87)$$

The position of the parabolas defined by Eq. 3.87 in the $\Delta X,\Delta Y$ plane is shown in Fig. 3.21. From Eq. 3.87 and Fig. 3.21, it is seen that the $\Delta Y'$ coordinate of the center of the parabola in the $\Delta X',\Delta Y'$ rotated coordinate system is not dependent on the parameter ω_1.

The only other parameter in the mapping of the plant pole variation is P_1. The qualitative effect of P_1 on the mapping of the plant pole variation is shown in Fig. 3.23. The plant pole variation used in the

FIG. 3.22 PLANT POLE VARIATION REGION USED TO ILLUSTRATE THE EFFECT OF P_1 ON THE MAPPING INTO THE ΔX-ΔY PLANE

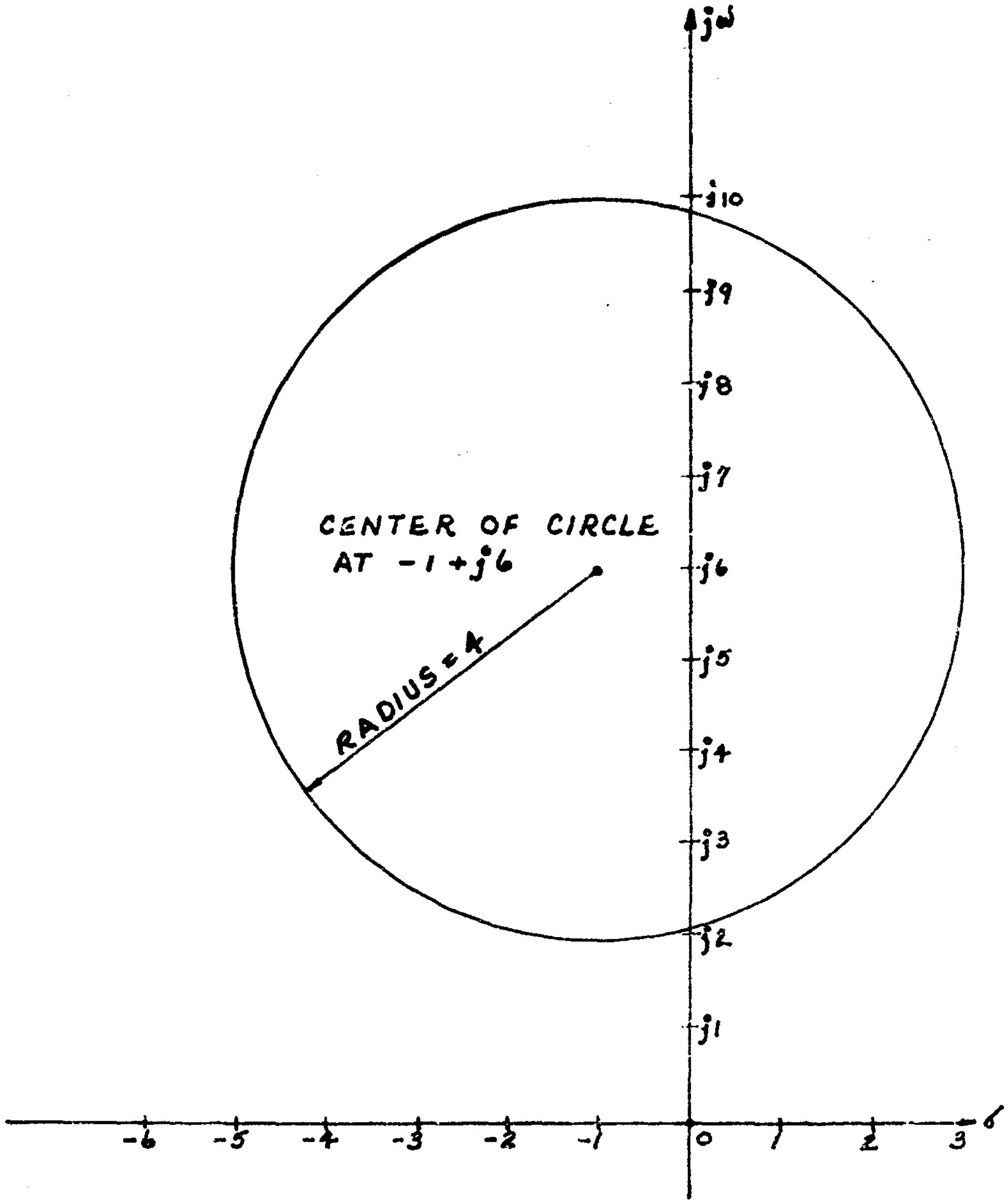
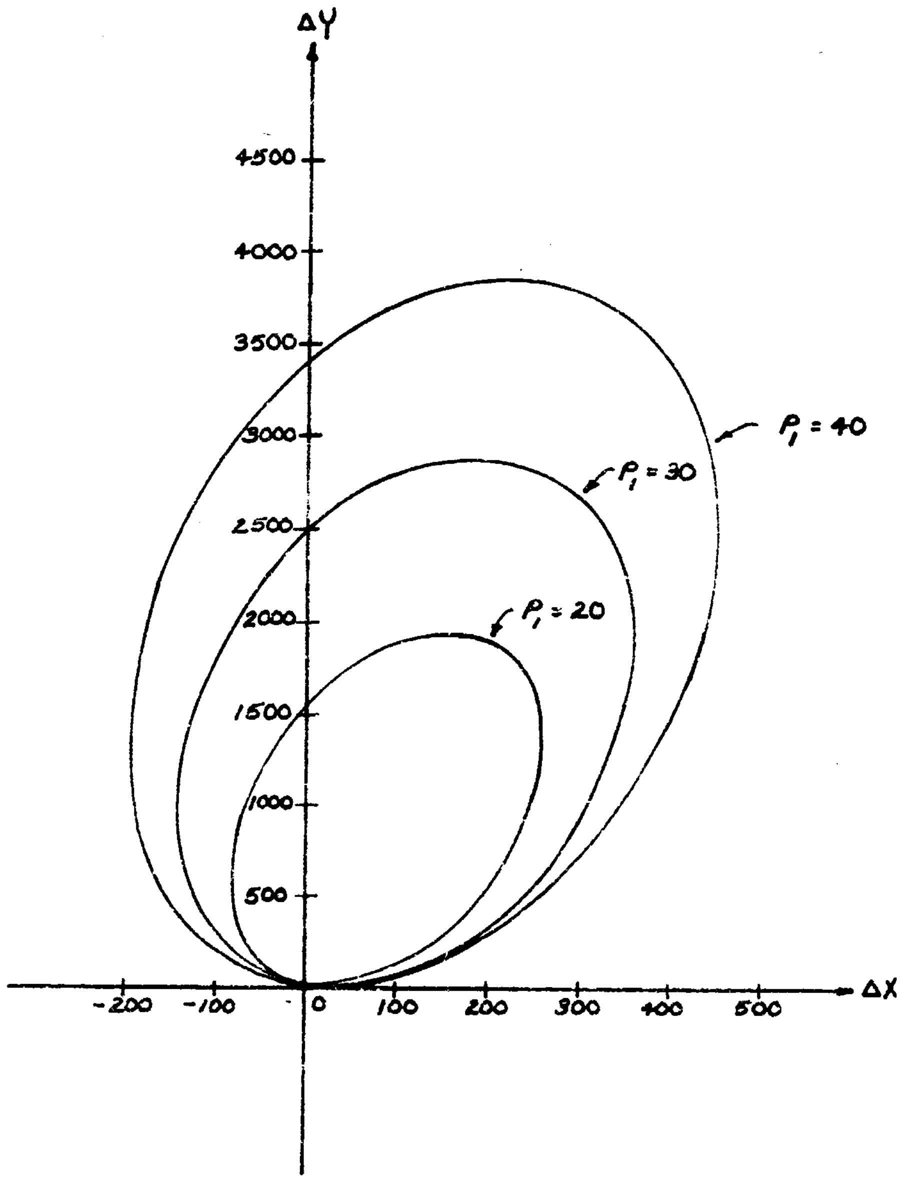


FIG. 3.23 MAPPING OF THE PLANT POLE VARIATION REGION AS A FUNCTION OF THE OPEN LOOP POLE P_1



mapping is shown in Fig. 3.22. From Fig. 3.23, it is observed that the size of the mapping of the plant pole variation increases with increasing P_1. Since the size of the mapping of the dominant closed loop pole region is not highly sensitive to P_1 (see Fig. 3.14), it is very important to bring P_1 in as far as possible to minimize the needed system gain kh.

3.10 The Effect of Plant Gain Variation on the Design

It is possible that the dominant closed loop poles may lie within their acceptable region for $k=k_{min}$ and yet move outside the acceptable region for $k>k_{min}$. Figure 3.24 illustrates this possibility. This undesirable possibility can be predicted by considering the angle of departure of the root locus from the dominant closed loop pole at $k = k_{min}$ and the angle of entry of the root locus into the compensation zeroes. This is the procedure used by Horowitz in his paper.[4]

Figure 3.25 illustrates the dominant closed loop poles within their acceptable region at $k = k_{min}$. For a given plant pole variation, the dominant closed loop poles for $k = k_{min}$ may lie anywhere on a boundary such as the boundary ABCD shown in Fig. 3.25. It is usually sufficient to check the angle of departure of the root locus for a few points around the boundary at $k = k_{min}$.

FIG. 3.24 POSSIBLE EFFECT OF GAIN VARIATION ON THE DOMINANT CLOSED LOOP POLES

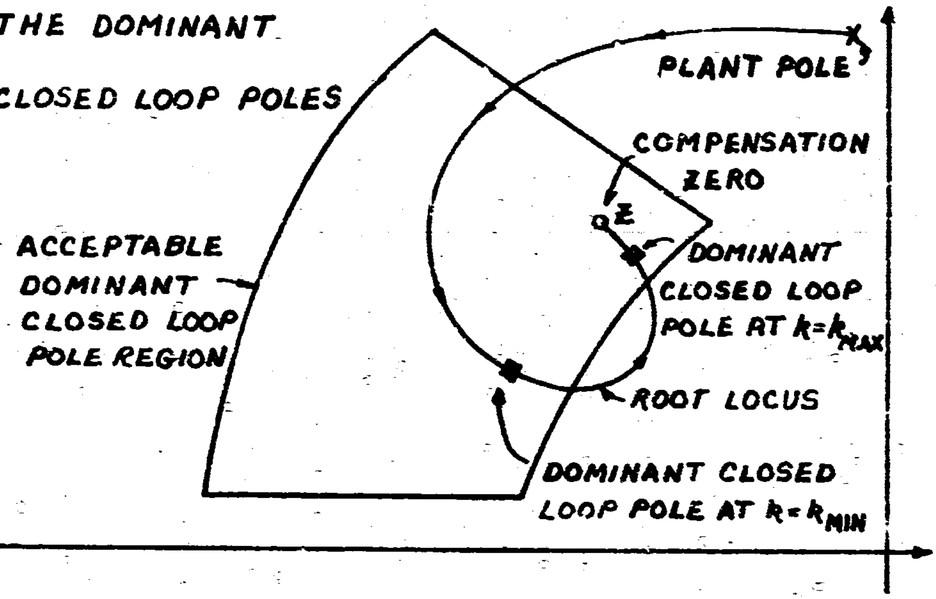
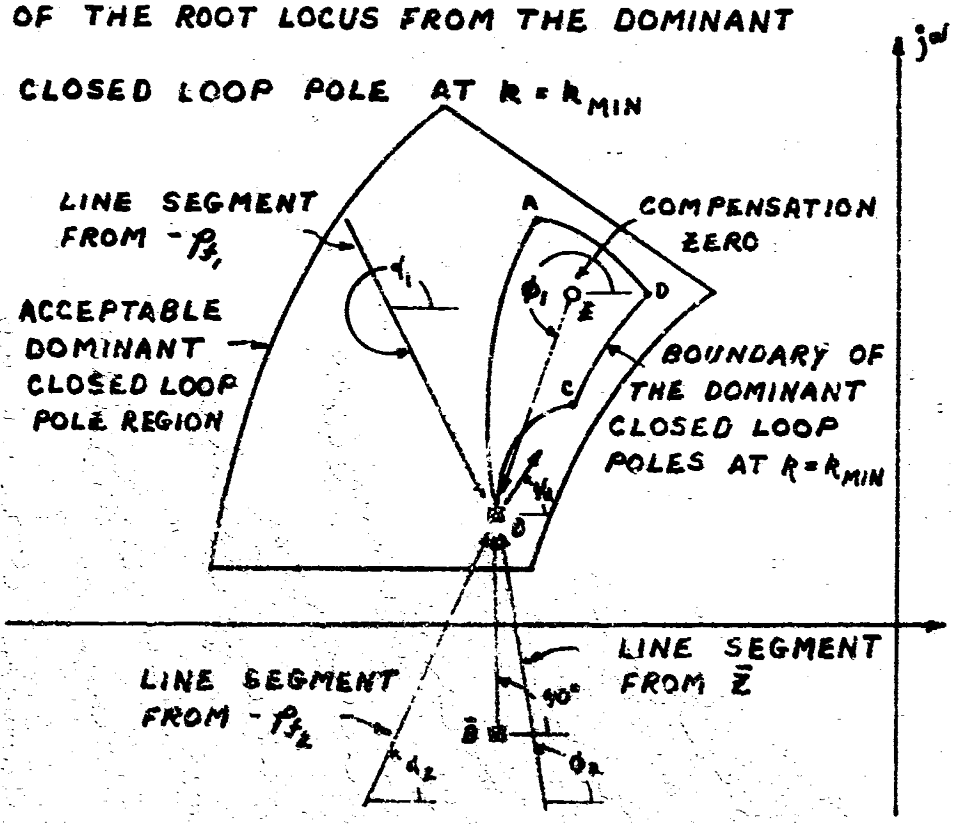


FIG. 3.25 COMPUTATION OF THE ANGLE OF DEPARTURE OF THE ROOT LOCUS FROM THE DOMINANT CLOSED LOOP POLE AT $k = k_{MIN}$



To outline the procedure, consider dominant closed loop poles at B and $\bar{B}$ for $k = k_{min}$ and "far-off" closed loop poles at $-p_{f_1}$ and $-p_{f_2}$ for $k = k_{min}$. Define the following angles

$$\angle \bar{B}\, B = 90° \tag{3.90}$$

$$\angle -p_{f_1} B = \alpha_1 \tag{3.91}$$

$$\angle -p_{f_2} B = \alpha_2 \tag{3.92}$$

$$\angle Z\, B = \Phi_1 \tag{3.93}$$

$$\angle \bar{Z}\, B = \Phi_2 \tag{3.94}$$

The angle of departure of the root locus ϕ_d from the dominant closed loop pole located at B is then

$$\phi_d = 180° - (\alpha_1 + \alpha_2 + 90° - \Phi_1 - \Phi_2) \tag{3.95}$$

It is relatively easy to ascertain for a particular mapping what angles of departure may lead to an unsatisfactory design. For example, for a dominant closed loop pole located at point B in Fig. 3.25, an angle of departure ϕ_d of -10° would probably be unsatisfactory.

If there is still some doubt whether the root locus remains within the acceptable region for a particular dominant closed loop pole location, the angle of entry of the root locus into the compensation zeroes

can be checked. As before consider dominant closed loop poles located at B and $\overline{B}$ and define the following angles

$$\angle \overline{Z}\, Z = 90° \tag{3.96}$$

$$\angle B\, Z = \theta_1 \tag{3.97}$$

$$\angle \overline{B}\, Z = \theta_2 \tag{3.98}$$

$$\angle -p_{f_1} Z = \theta_3 \tag{3.99}$$

$$\angle -p_{f_2} Z = \theta_4 \tag{3.100}$$

The angle of entry of the root locus ψ_e to the complex zero Z for dominant closed loop poles located at B and $\overline{B}$ is then

$$\psi_e = \theta_1+\theta_2+\theta_3+\theta_4-90°-180° \tag{3.101}$$

As stated for the angle of departure criteria, it should be relatively easy to determine for a given mapping what angles of entry are acceptable. Considering the case again for dominant closed loop poles located at B and $\overline{B}$, if the root locus had an angle of departure from B of -10° and an angle of entry to the compensation zero Z of +10°, this would probably confirm that the root locus is outside the dominant closed loop pole region for some value of $k > k_{min}$.

If such a situation arises in the design, the procedure is to increase the value of added gain k until all angles of departure and entry are satisfactory. The object, of course, is to obtain a design with the least amount of system gain kk.

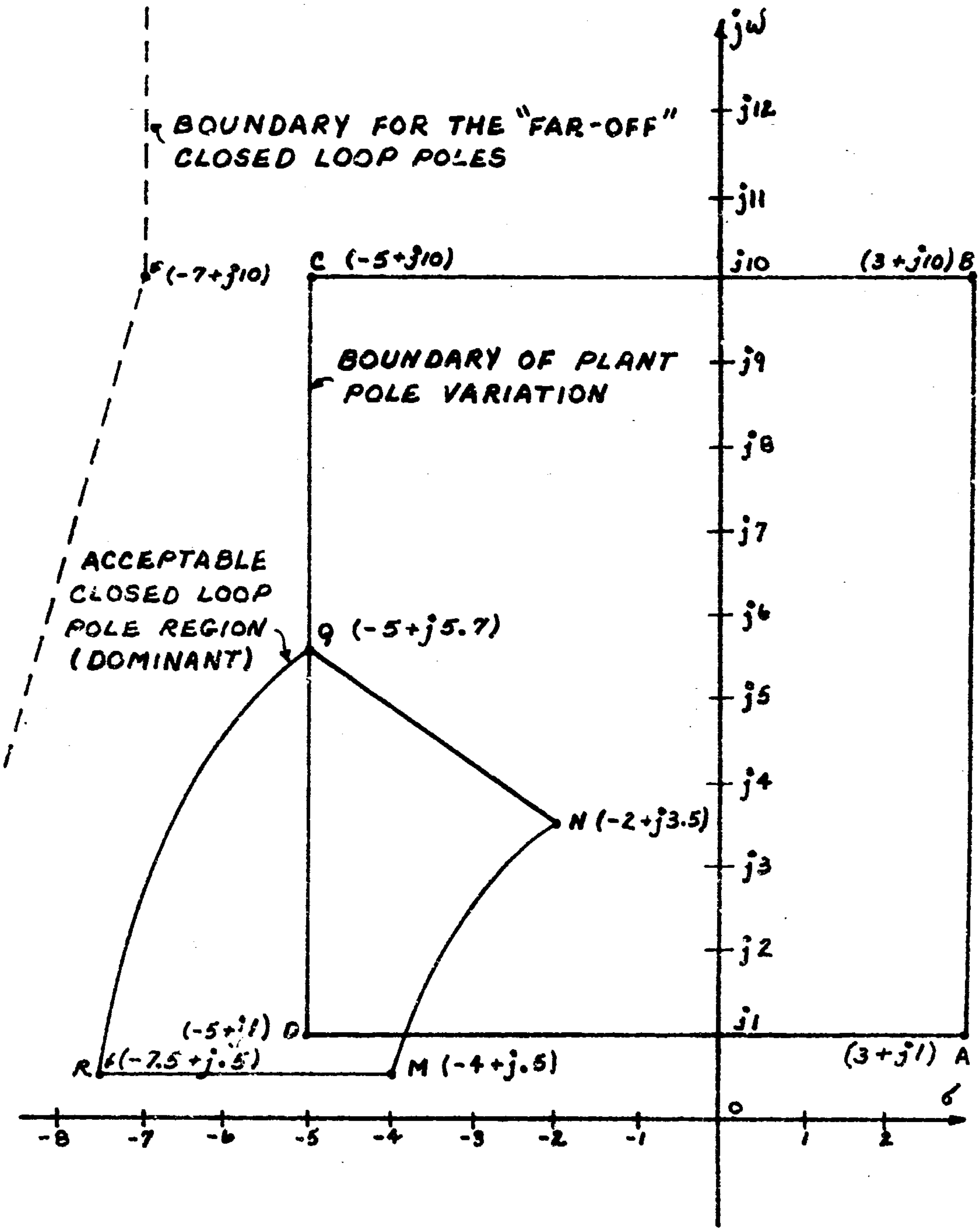
3.11 Design Example

A design example is presented in this section based on the previous design procedure. The region of plant pole variation, acceptable dominant closed loop pole region and boundary for the "far-off" closed loop poles are shown in Fig. 3.26. The acceptable dominant closed loop pole region is the same as used in the design example of Chapter II. The plant pole variation is essentially the same as used by Horowitz.[4] The boundary for the "far-off" closed loop poles has been arbitrarily chosen as shown in Fig. 3.26.

Following the design procedure outlined in Section 3 of this Chapter, the dominant closed loop pole region is mapped into the X,Y plane with parameter P_1, γ and S_p.

After several computor runs, values for P_1 and γ of 30 and 20,000 respectively were used for the first design. The value of 30 for P_1 was chosen because this value places the "far-off" closed loop poles very near the boundary shown in Fig. 3.26. Using Eq. 3.18 with

FIG. 3.26 S-DOMAIN SPECIFICATIONS FOR THE DESIGN EXAMPLE



P_r = 60 and $p_{f_1} = \overline{p}_{f_2}$ = +15-j10, a value of approximately 19,000 for γ is obtained which is quite close to the value used in the first design. The value of P_r for this approximation was found from the acceptable dominant closed loop pole region while the values of p_{f_1} and p_{f_2} were estimated from the knowledge of the anticipated root locus. The values of S_p used were -o, 2 and 10 which correspond to the minimum, median and maximum possible values of S_p for the plant pole variation shown in Fig. 3.26. As anticipated the mapping of dominant closed loop pole region is not highly sensitive to the value of S_p at this large value of γ. The mapping of the dominant closed loop pole region for these parameter values is shown in Fig. 3.27.

The mapping of the plant pole variation for various values of P_1 including P_1=30 is shown in Fig. 3.28.

It should be emphasized that the only practical means of performing these mapping operations is on a digital computor. On such a machine, the mapping of the dominant closed loop pole region may be easily performed for many different combinations of γ and P_1. The same values of P_1 are then used in the mapping of the plant pole variation. Once the mappings have been plotted, it is not difficult to choose a minimum value

FIG. 3.27 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION FOR THE DESIGN EXAMPLE

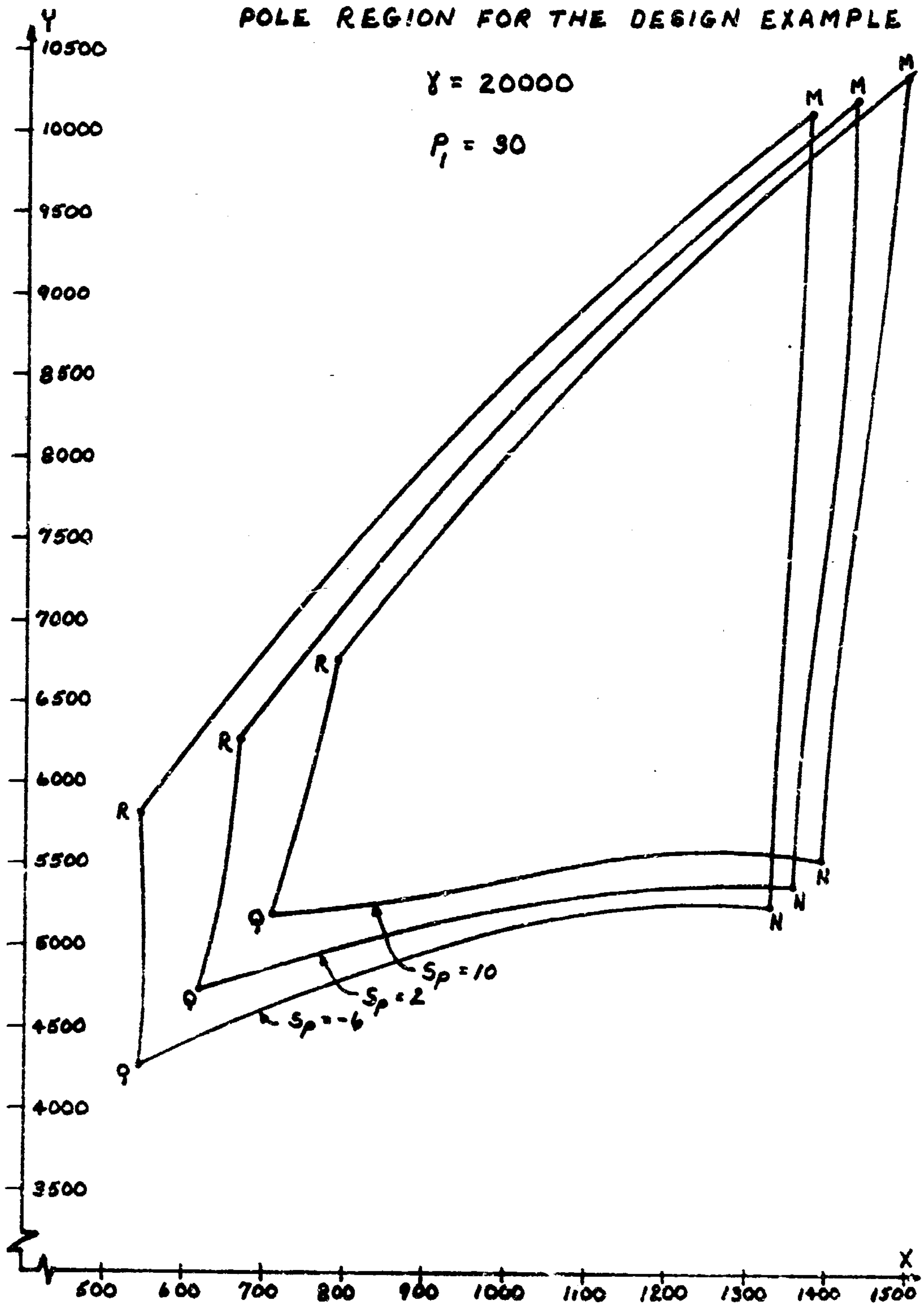
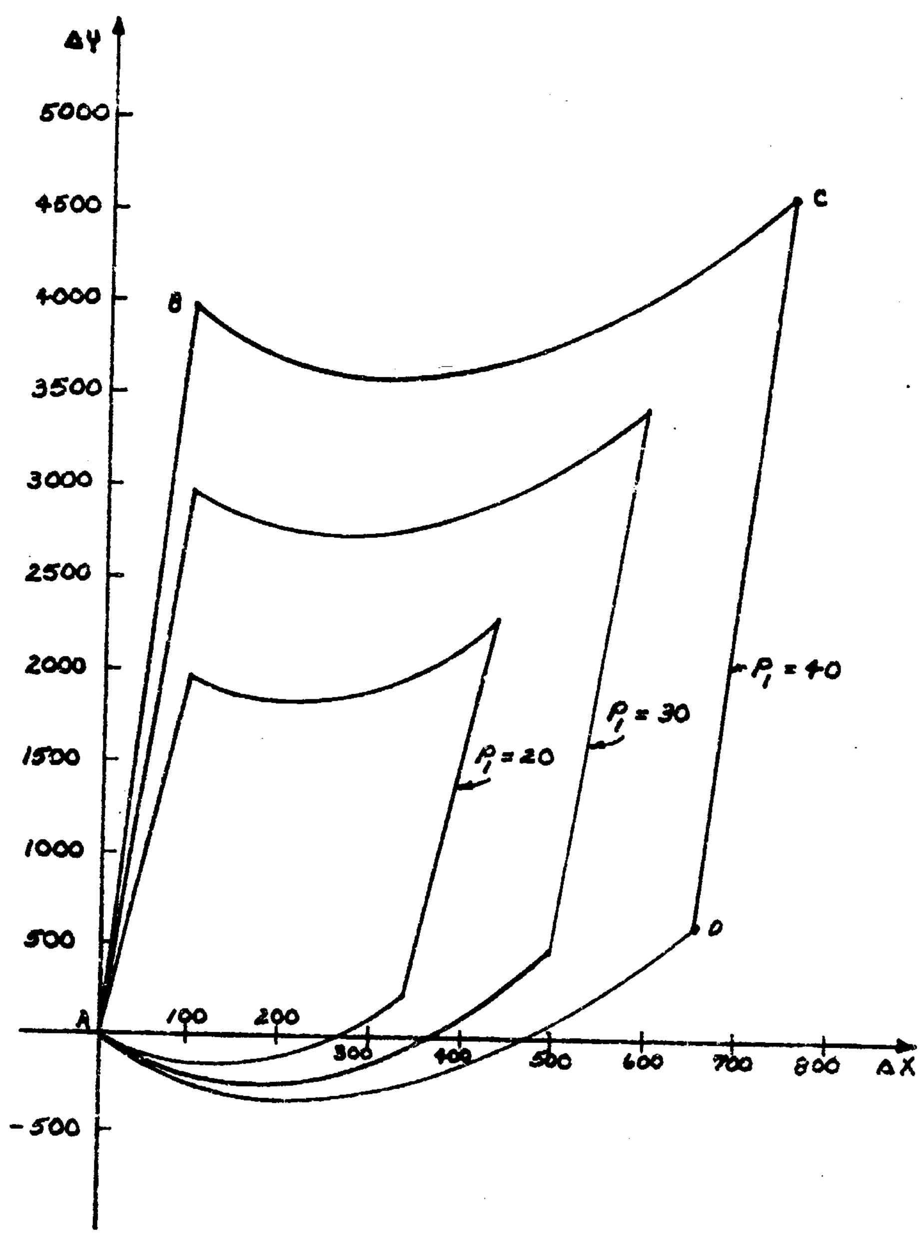


FIG. 3.28 MAPPING OF THE PLANT POLE VARIATION FOR THE DESIGN EXAMPLE



of P_1 such that the "far-off" closed loop poles are at the vertical boundary shown in Fig. 3.26 and a value of γ such that the mapping of the plant pole variation may be fitted inside the mapping of the dominant closed loop pole region for the same value of P_1.

The mapping of the dominant closed loop pole region with the mapping of the plant pole variation placed inside it is shown in Fig. 3.29. Since the mapping of the plant pole variation does not quite fit inside the mapping of the dominant closed loop pole region, a slightly larger value of γ may have to be used.

The values of kK, S_o and P_o are solved for using Eqs. 3.31, 3.32 and 3.33 and point A in Fig. 3.29. At point A

$$X_a = 800 \qquad Y_a = 5450$$

and from Fig. 3.26, at point A

$$S_{p_a} = -6 \qquad P_{p_a} = 10$$

For $\gamma = 20000$ and $P_1 = 30$, kK, S_o and P_o are

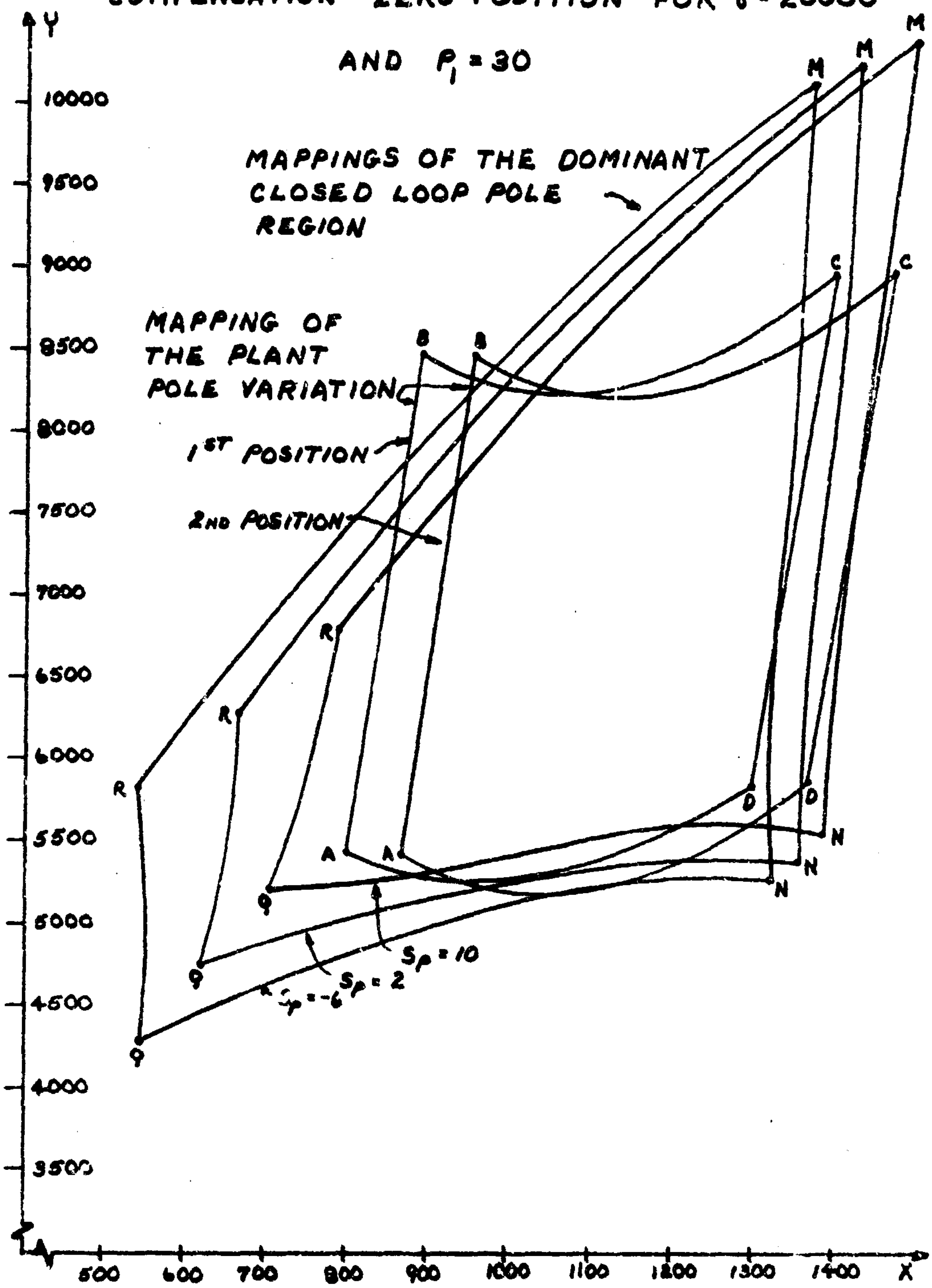
$$kK = X_a - S_{p_a} P_1 - P_{p_a}$$

$$= 800-(-6)(30)-10 = 970$$

$$S_o = \frac{Y_a - P_{p_a} P_1}{kK}$$

$$= \frac{5450-10(30)}{970} = 5.32$$

FIG. 3.29 CALCULATION OF SYSTEM GAIN AND COMPENSATION ZERO POSITION FOR $\gamma = 20000$ AND $P_1 = 30$



$$P_o = \frac{\gamma}{kK} = \frac{20000}{970} = 20.61$$

Solving for the position of the compensation zeroes yields

$$S_o = -2\alpha_z$$

$$\rightarrow \alpha_z = \frac{S_o}{-2} = -2.66$$

$$P_o = \alpha_z^2 + \omega_z^2$$

$$\rightarrow \omega_z = \sqrt{P_o - \alpha_z^2} = \sqrt{20.61-5.32}$$

$$= 3.60$$

The position of the compensation zeroes Z, $\overline{Z}$ is then

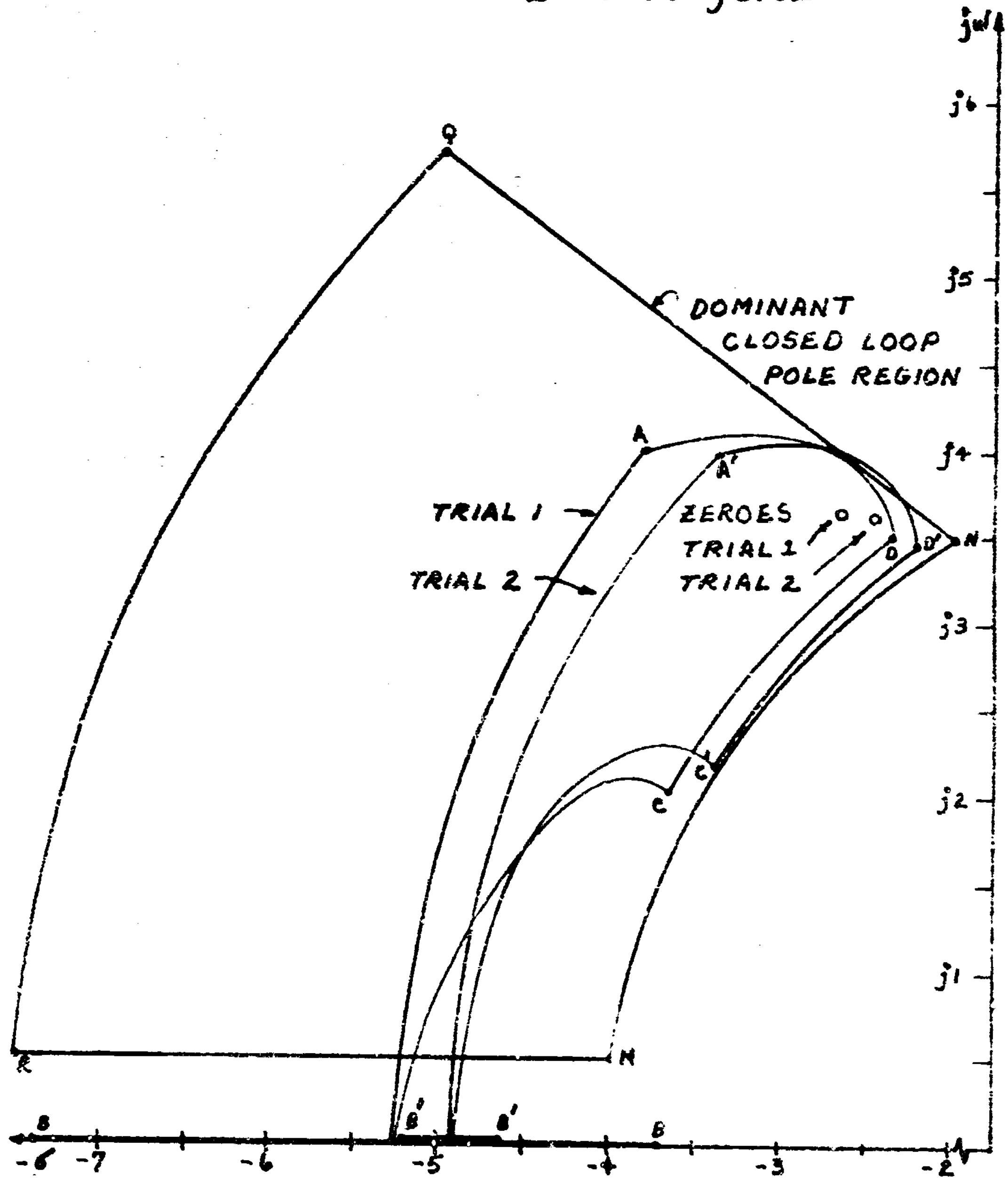
$$Z, \overline{Z} = -2.66 \pm j3.60$$

The actual closed loop poles for this choice of system gain and compensation zero position are found by determining the roots of the characteristic equation $1+L_d(s) = 0$ where $L_d(s)$ is given by Eq. 1.12 for points around the boundary of the plant pole variation. The convergence procedures in Appendix A were used to factor the resulting fourth order polynomial. The results of this operation are shown in Fig. 3.30 (Trial 1). The points A, B, C and D correspond with those in Fig. 3.26. From Fig. 3.30, it is seen that the dominant closed loop poles lie outside

FIG. 3.30 DOMINANT ROOT TEST FOR γ = 20000 AND P_1 = 30

TRIAL 1 : kK = 970   z = -2.66 + j 3.60

TRIAL 2 : kK = 1045   z = -2.46 + j 3.62



their acceptable region for plant poles located at points B,$\bar{B}$. The closest approach of the "far-off" closed loop poles also occurs when the plant poles are at points B,$\bar{B}$ which is -6.498 ± j26.395. For all other points used in the dominant root test, the "far-off" closed loop poles were well to the left of the established boundary shown in Fig. 3.26.

In an attempt to place the dominant closed loop poles within their acceptable region, the position of the mapping of the plant pole variation within the mapping of the dominant closed loop pole region was slightly altered as shown in Fig. 3.29. For this new position and using the new coordinates for point A, the values of system gain and compensation zero position are

$$kK = 1045 \qquad Z,\bar{Z} = -2.46 \pm j3.62$$

The results of this trial are shown in Fig. 3.30 (Trial 2). The dominant closed loop poles are still outside their acceptable region when the plant poles lie at points B, $\bar{B}$. The closest approach of the "far-off" closed loop poles is -7.019 ± j27.571 when the plant poles are at points B,$\bar{B}$. Therefore the dominant closed loop pole region must be mapped into the X,Y plane using a larger value of γ. The mapping of the plant

pole variation for $P_1 = 30$ in the ΔX, ΔY plane remains unchanged.

Figure 3.31 shows the mapping of the dominant closed loop pole region for $\gamma = 22000$ with the mapping of the plant pole variation fitted within the interior. Using point A again gives the following

$$kK = 1165 \qquad Z,\overline{Z} = -2.432 \pm j3.608$$

From Fig. 3.32, this value of gain and compensation zero position place the dominant closed loop poles within their acceptable region for all plant pole positions. The closest approach of the "far-off" closed loop poles occurs again when the plant poles are at $B,\overline{B}$ and is $-7.313 \pm j29.661$.

From inspection of Figs. 3.22 and 3.23, it is noted that the mapping of the plant pole variation is quite sensitive to the value of P_1 and that the size of the mapping of the plant pole variation decreases with decreasing P_1. Therefore it may be possible to place the dominant closed loop poles within their acceptable region at a slightly smaller value of system gain kK if the value of P_1 is decreased. This will also place the "far-off" closed loop poles closer in. Figure 3.33 shows the results of using a value of $P_1 = 28$ and the same of gain and zero position as for Trial 2 with $\gamma = 20000$. The saving in gain between

FIG. 3.31 CALCULATION OF SYSTEM GAIN AND COMPENSATION ZERO LOCATION FOR $\gamma = 22000$ AND $P_1 = 30$

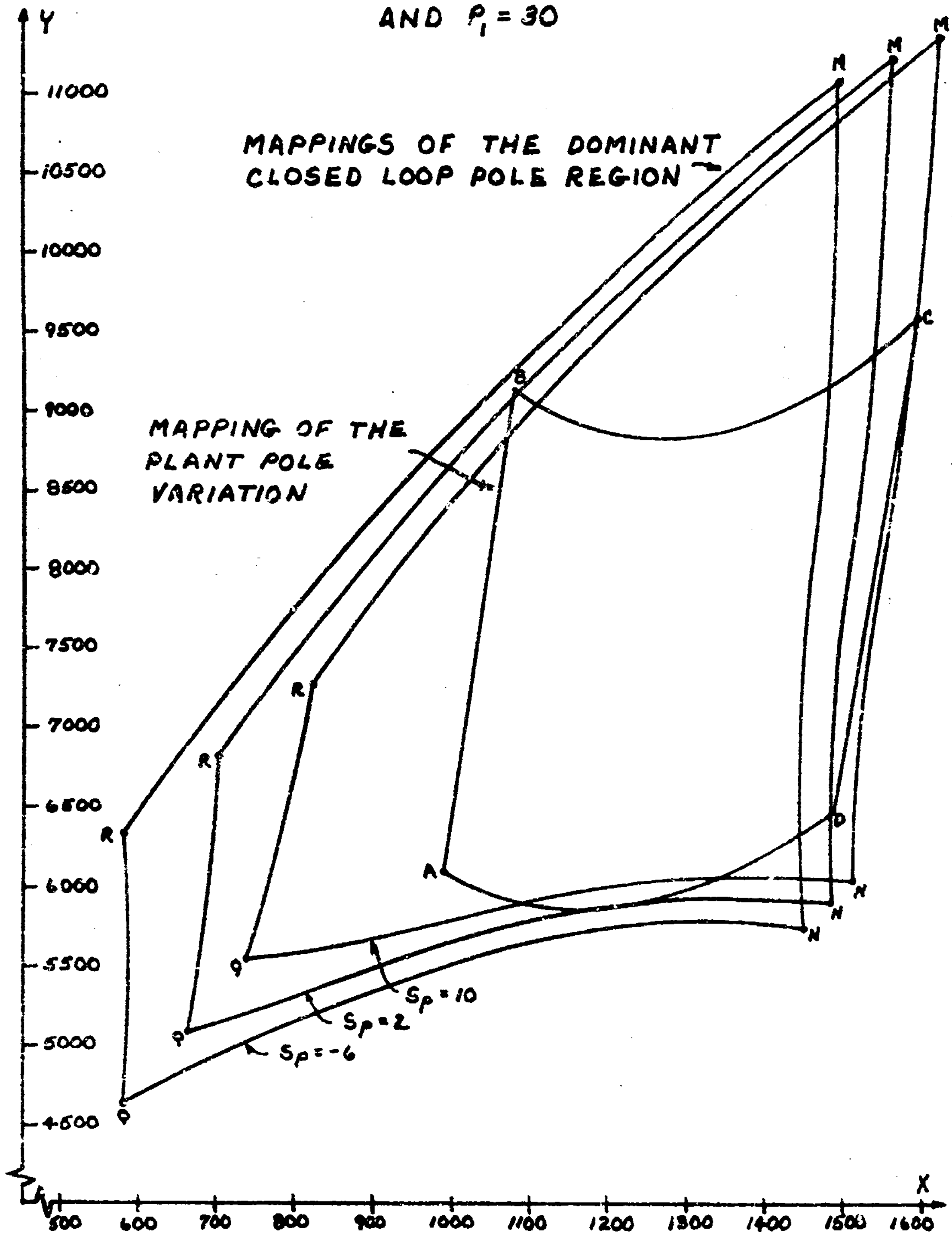


FIG. 3.32 DOMINANT ROOT TEST FOR $\gamma = 22000$ AND $P_1 = 30$

$RK = 1165 \quad Z = -2.50 + j3.56$

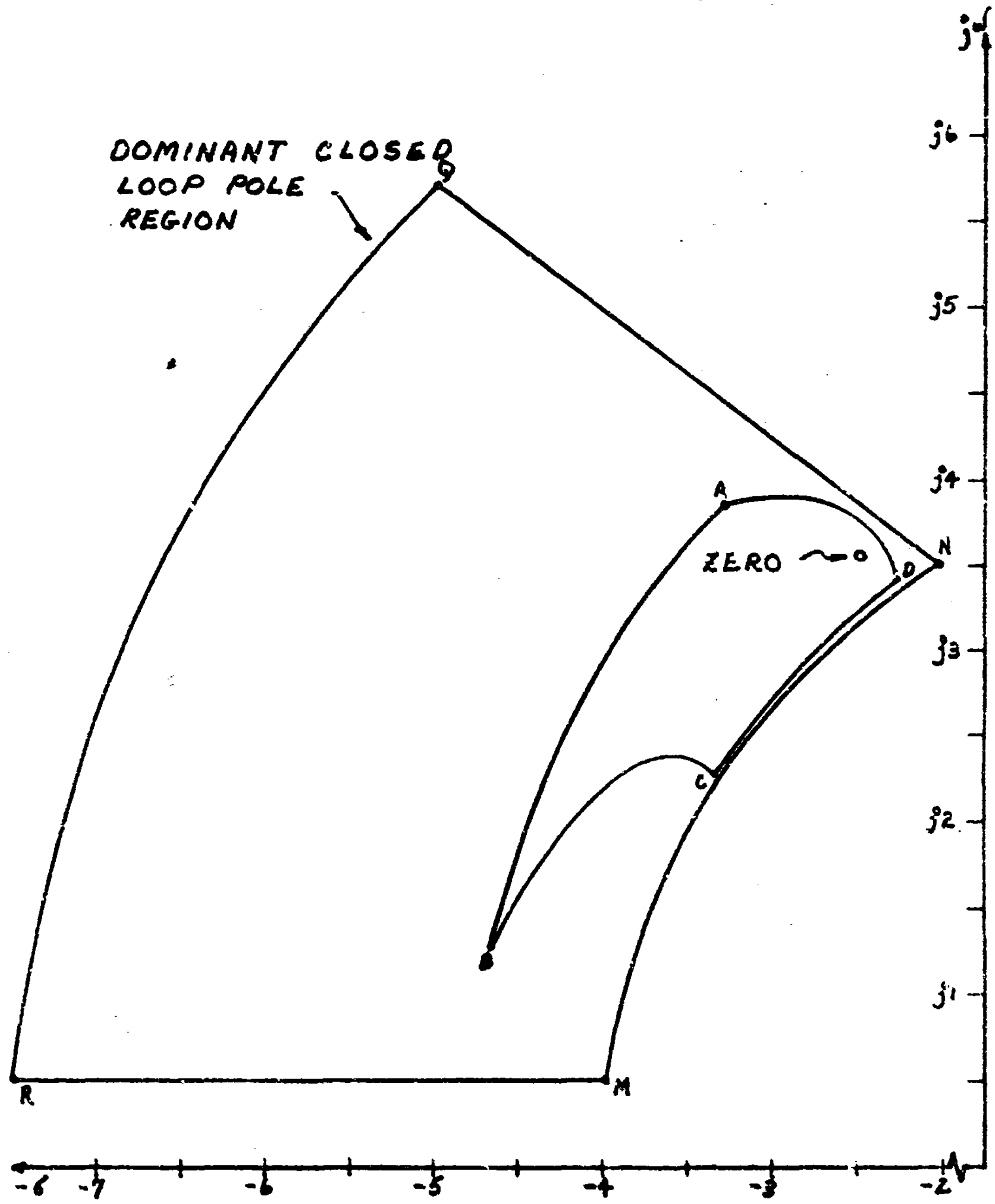
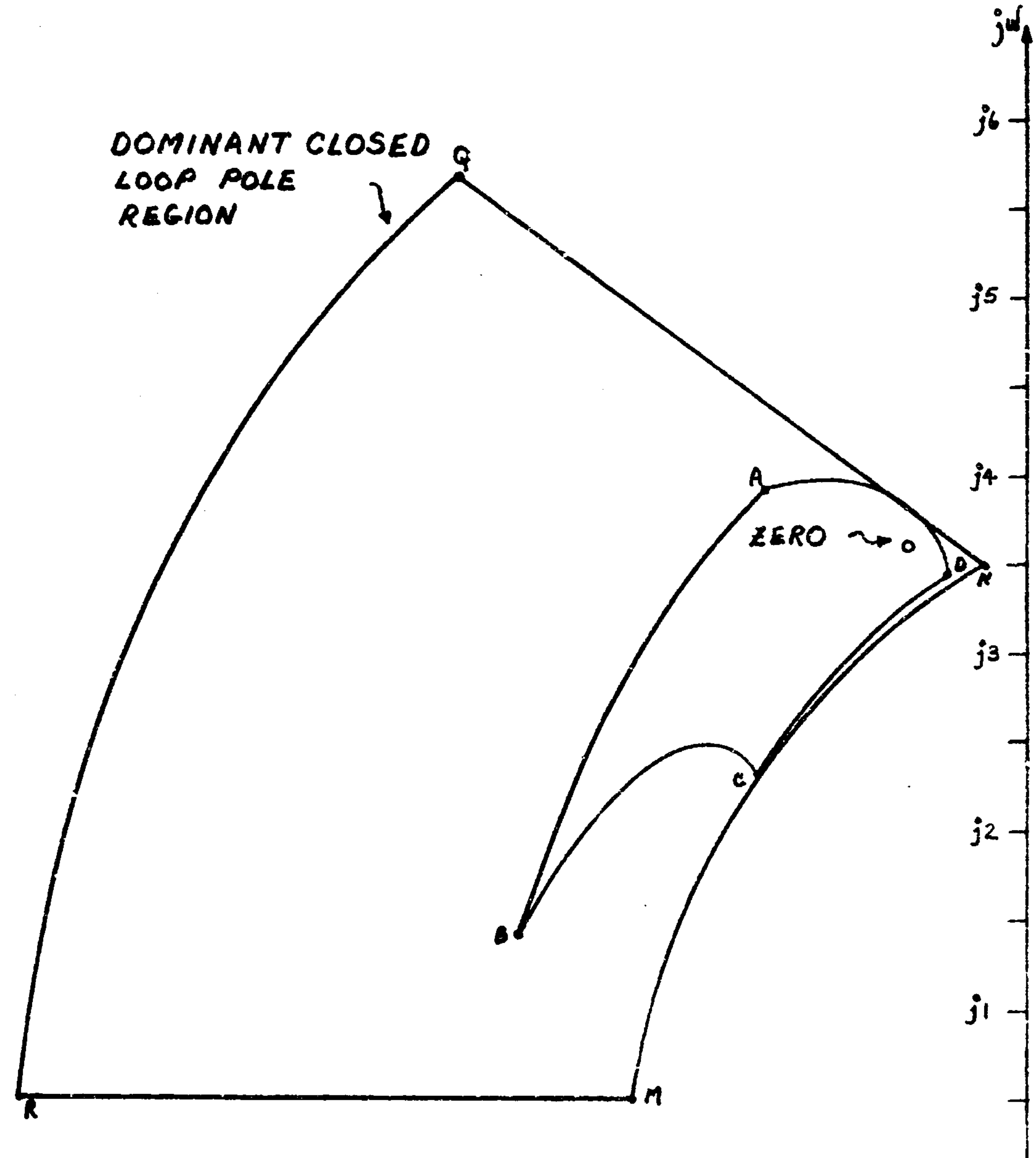


FIG. 3.33 DOMINANT ROOT TEST FOR $\gamma = 20000$ AND $P_1 = 28$

$RK = 1045 \quad Z = -2.46 + j\,3.62$



kK = 1045 and kK = 1165 is approximately 1 db. The closest approach of the "far-off" closed loop poles is $-6.324 \pm j28.355$ with the plant poles at $B,\bar{B}$. This design is probably satisfactory since the closed loop poles lie to the left of the boundary shown in Fig. 3.26 for all but three other plant pole positions near point B used in the root test.

From Figs. 3.29 and 3.31, it is noted that the mapping of the plant pole variation need not fit completely inside the mapping of the dominant closed loop pole region for all values of S_p in order for the dominant closed loop poles to lie within their acceptable region. The requirement that must be satisfied is that the mapping of $S_i = S_{p_i}$ in the $\Delta X, \Delta Y$ plane must fit inside the mapping of the dominant closed loop pole region for $S_p = S_{p_i}$. From Fig. 3.31, the line segment $\overline{AB}$ corresponds to $S_p = -6$ which is completely within the mapping of the dominant closed loop pole region for $S_p = -6$. The line segment $\overline{CD}$ corresponds to $S_p = 10$ which is completely within the mapping of the dominant closed loop pole region for $S_p = 10$.

So for in the design example, the problem of variation in the plant gain factor has not been considered. The actual value of gain which must be added to the system is given by Eq. 3.34. For the

design of $P_1 = 28$, $Z,\overline{Z} = -2.46 \pm j3.62$ and $kK = 1045$, the value is $1045/k_{min}$. If the plant gain variation is assumed to be

$$k_{min} = 1 \leq k \leq 1000 = k_{max}$$

the compensation networks must have a gain of at least 1045. At $k = k_{max}$, the system gain is 1.045×10^6 and for this large value of gain, the dominant closed loop poles are essentially at the position of the compensation zeroes. Using the method outlined in Section 10 of this chapter, the angles of departure of the root locus for dominant closed loop poles located at points A, B, C and D in Fig. 3.33 were computed. The results were as follows:

Point A: $\psi_d = -14°$

B: $\psi_d = 69°$

C: $\psi_d = 64°$

D: $\psi_d = 133°$

These values are certainly satisfactory for the acceptable dominant closed loop pole region shown in Fig. 3.33. Therefore the design is complete.

3.12 Summary

The design procedure for gain and plant pole variation is summarized below.

1.) Map the acceptable dominant closed loop pole region into the X,Y plane using

Eqs. 3.14 and 3.15 with parameters γ, P_1 and S_p. Map the plant pole variation into the ΔX, ΔY plane using Eqs. 3.16 and 3.17 with parameter P_1. Determine the value of γ and P_1 such that the "far-off" closed loop poles just satisfy the minimum damping factor specifications for the problem and the mapping of the plant pole variation in the $\Delta X, \Delta \dot{Y}$ plane can be fitted into the interior of the mapping of the dominant closed loop pole region in the X,Y plane.

2.) Solve for the values of kK, S_o and P_o using Eqs. 3.11, 3.12 and 3.13 which determine the value of system gain and the compensation zero position.

3.) Check the design by determining the actual closed loop poles for plant poles lying on the boundary of the plant pole variation. This is accomplished by determining the roots of $1+L_d(s) = 0$. Check the final design for plant gain variation by using the method outlined in Section 10 of this chapter.

This design procedure which includes the effect of the "far-off" pole P_1 has several advantages over the third order system considered by Horowitz.[4] Using the third order approximation, the "far-off" closed loop poles can not be positioned so that the damping factor specifications are just satisfied. This results in a waste of system gain since the "far-off" open loop pole must then be placed sufficiently "far-off" so that its effect on the dominant closed loop poles is negligible. In the fourth order approximation, the effect of this "far-off" pole is considered on both the mapping of the dominant closed loop pole region and plant pole variation and can be used to determine a more economical design.

The next chapter considers the additional effect of a drifting zero on the real axis on the design procedure.

CHAPTER IV

PROBLEM OF SIMULTANEOUS PLANT GAIN, POLE AND ZERO VARIATION

4.1 Problem Definition

This chapter presents an approximate design procedure for handling the added problem of a drifting zero on the real axis. This zero can be considered to be part of a plant with a transfer function P(s), given by

$$P(s) = \frac{k(s+z)}{s(s^2+S_p s+P_p)} \qquad (4.1)$$

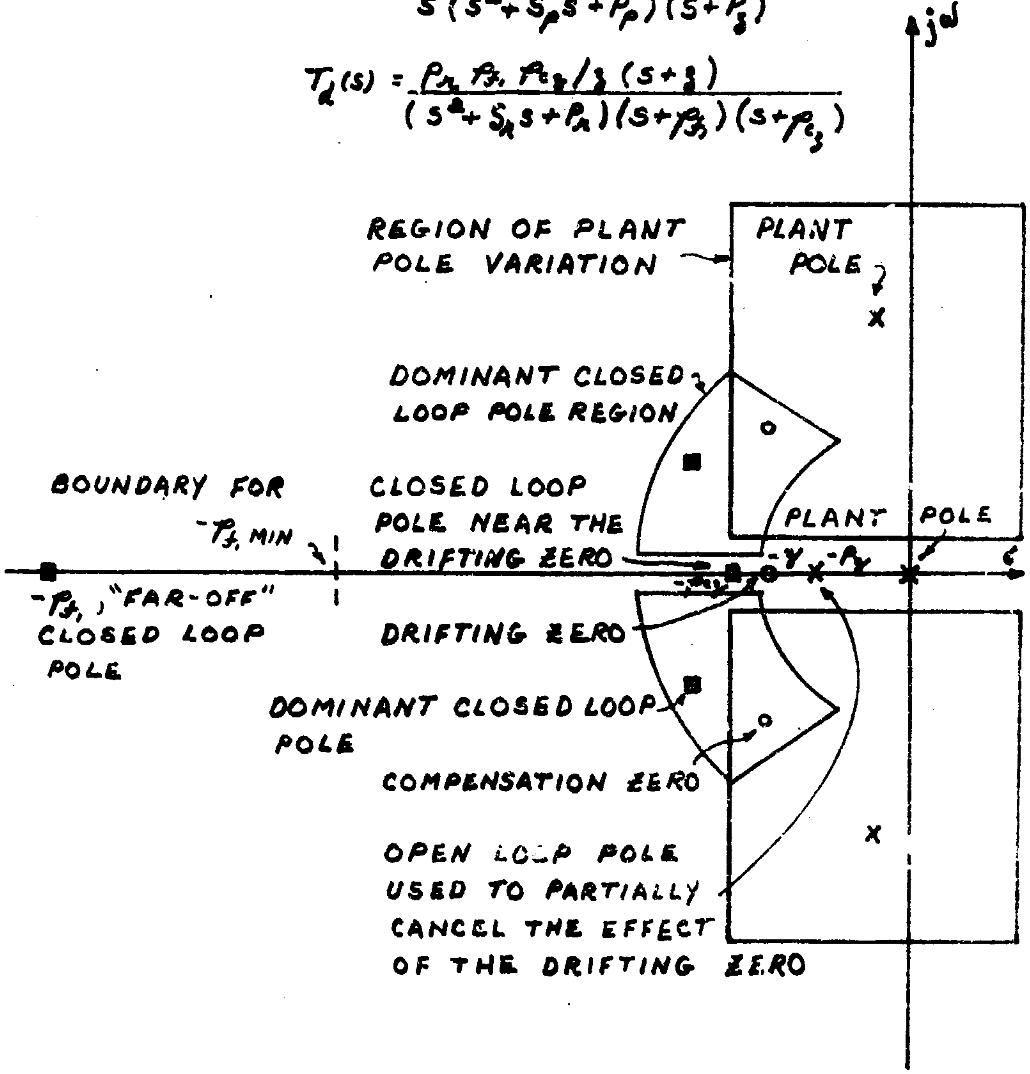
where -z is the position of the drifting zero on the real axis. Typically this zero on the real axis is close to the origin and hence to the dominant closed loop poles, and has an appreciable ffect on the position of the dominant closed loop poles. The problem is then to choose the compensation zero position and the value of system gain such that the dominant closed loop poles lie within their acceptable region despite variations in the plant zero, parameter z, and in the plant poles, parameters S_p and P_p. The problem is shown in Fig. 4.1.

FIG. 4.1 PROBLEM OF PLANT GAIN, POLE AND ZERO VARIATION

$$P(s) = \frac{k(s+\gamma)}{s(s^2+S_p s+P_p)}$$

$$L_d(s) = \frac{kK(s^2+S_c s+P_c)(s+\gamma)}{s(s^2+S_p s+P_p)(s+P_z)}$$

$$T_d(s) = \frac{P_n\, \beta_f\, P_{cz}/z\,(s+z)}{(s^2+S_n s+P_n)(s+\beta_f)(s+P_{cz})}$$



4.2 Design Philosophy

Any drifting zero of $P(s)$ will appear as a drifting zero of $T(s)$ (the system transmission), since it is impossible to precisely cancel such zeroes. To minimize the effect of a drifting zero on $T(s)$, a pole of $L_d(s)$ is placed near the zero in an attempt to at least partially cancel the effect of this zero. If the system gain is sufficiently high, a closed loop pole will be very close to the zero despite its drift. The resulting dipole will have a negligible effect on the system response, if the system gain is large enough.

4.3 The Effect of Zero Variation on the Dominant Closed Loop Poles

The effect of a pole-zero pair on the real axis, near the dominant pole region, is shown in Figs. 4.2 and 4.3. These figures are for a system designed according to the procedure in Chapter III, with the added pole-zero pair on the real axis. The mappings were obtained by factoring the roots of $1+L_d(s)=0$, where $L_d(s)$ is given by

$$L_d(s) = \frac{K(s^2+S_o s+P_o)(s+z)}{s(s^2+S_p s+P_p)(s+P_1)(s+P_z)} \qquad (4.2)$$

and z and P_z represent the pole-zero pair on the real axis. The region of plant pole variation is the same as was used in Chapter III, as is the acceptable dominant closed loop pole region.

FIG. 4.2 THE EFFECT OF ZERO VARIATION ON THE DOMINANT CLOSED LOOP POLES

(CASE 1)

PARAMETERS: KK = 1156 $P_1 = 30$

$Z = -2.50 + j3.56$ $\zeta = 3$ $P_\zeta = 2$

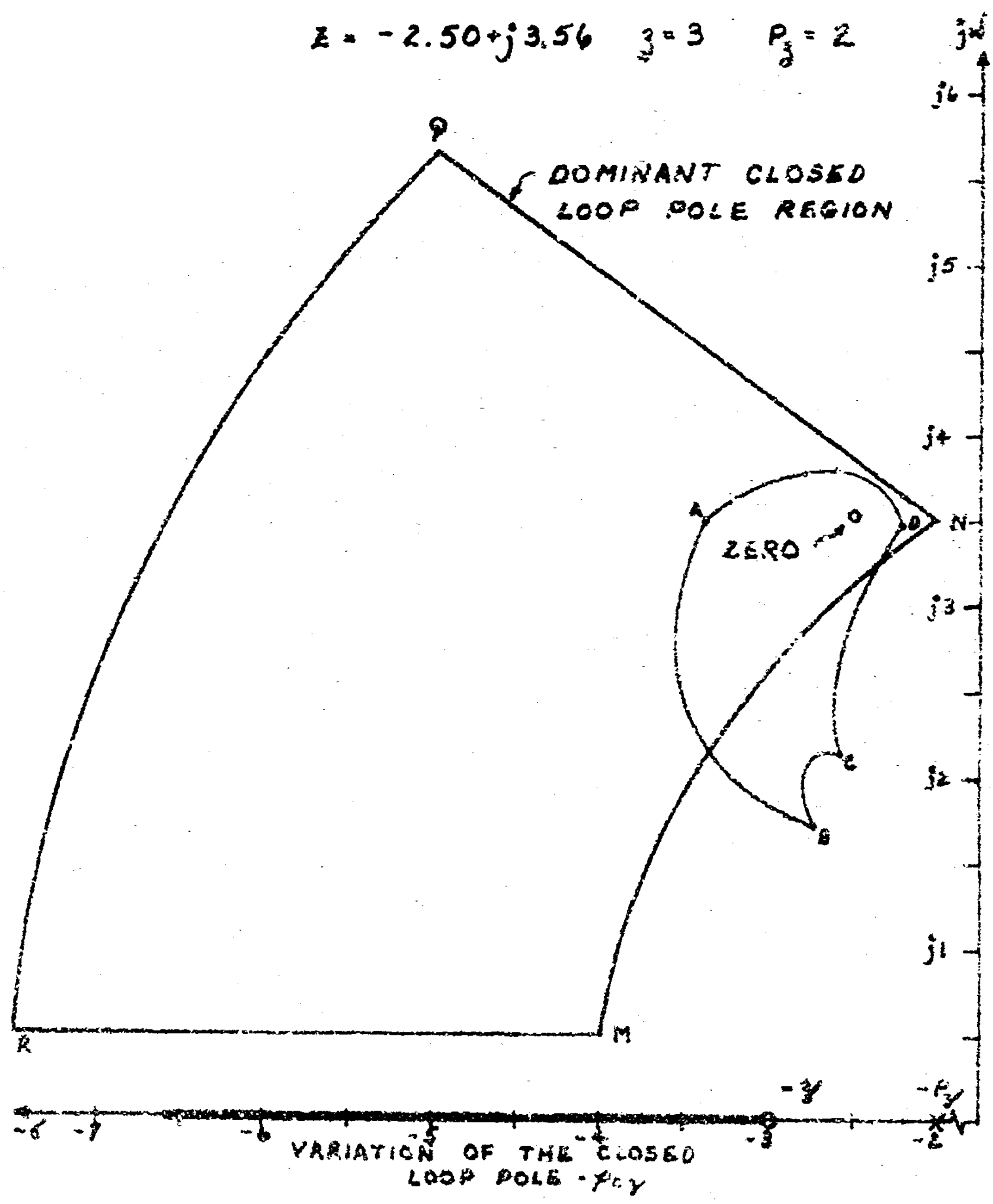


FIG. 4.5 THE EFFECT OF ZERO VARIATION ON THE DOMINANT CLOSED LOOP POLES

(CASE 2)

PARAMETERS: $kK = 1156$ $\rho_1 = 30$

$z = -2.50 + j3.56$ $\zeta = 1$ $\rho_2 = 2$

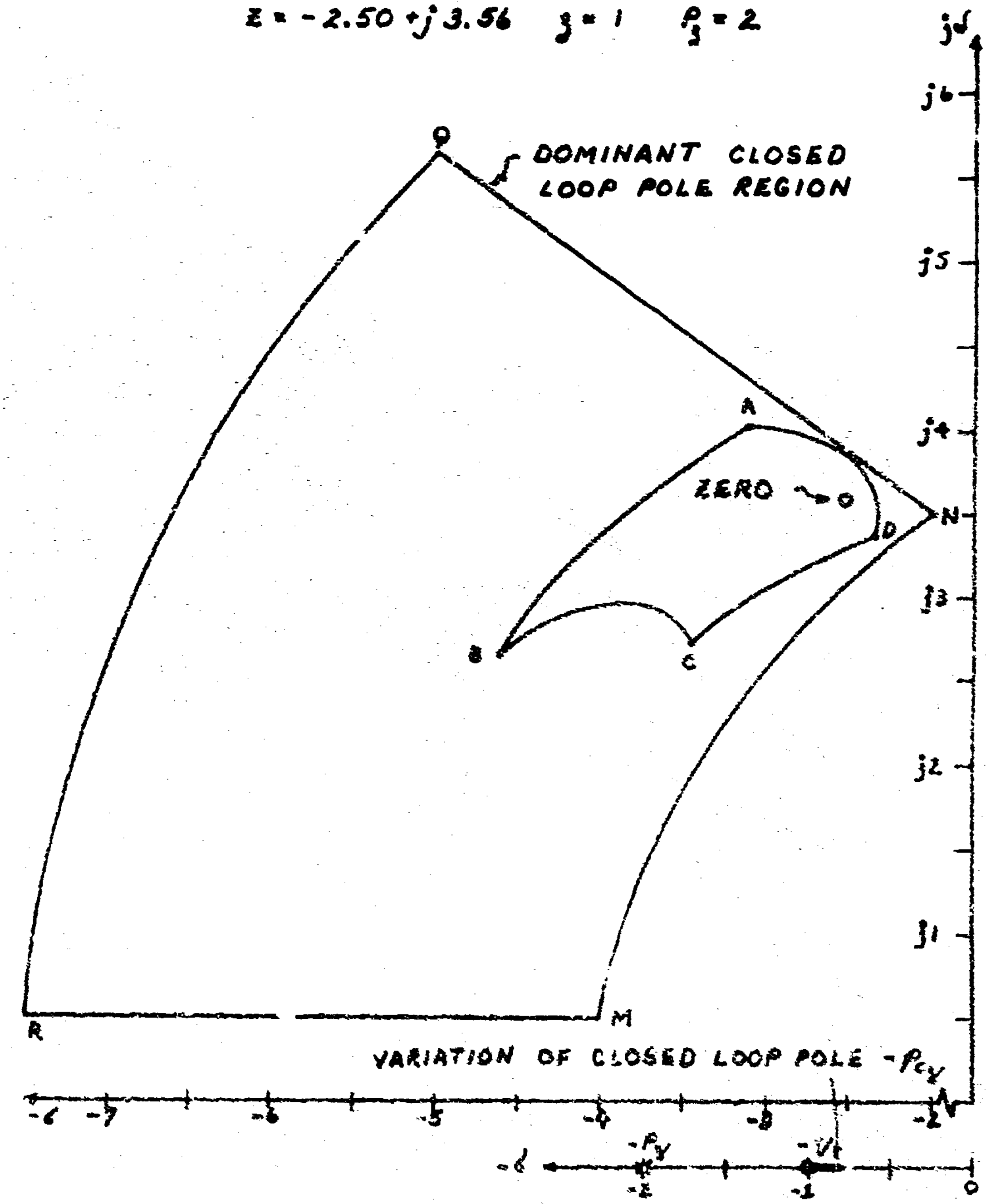


Figure 4.2 illustrates the effect on the dominant closed loop poles for $P_z=2$ and $z=3$; it is similar to the effect of lag compensation. Figure 4.3 is for $P_z=2$ and $z=1$ which has an effect similar to lead compensation. These two figures, then, illustrate the effect on the dominant closed loop poles of a zero drifting from -1 to -3 when an open loop pole of $L_d(s)$ is placed at -2. As seen from Fig. 4.2, the system designed according to the procedure in Chapter III is not adequate to handle this zero variation on the real axis, since the dominant closed loop poles lie outside their acceptable region. The dominant closed loop poles could be forced into their acceptable region by using a larger value of system gain. However, this may result in a waste of system gain. Horowitz presents a method[9] to determine whether a design for a given region of plant pole variation is adequate to handle the effect of a drifting zero on the real axis for a specified dipole (closed loop pole-zero) separation. The presence of the pole-zero pair on the real axis may influence the choice of the compensation zero position. The design procedure presented in this chapter takes into account the zero variation in determining the position of the compensation zeroes.

4.4 Design Equations

In considering this problem, it does not take long to determine that the drifting zero added to the plant pole variation, complicates the design procedure to a considerable extent. The dominant loop transmission must now be designed to take into account two independent types of variation, i.e., zero variation along the real axis and plant pole variation in the complex plane. Because of this, the nearest "far-off" open loop P_1 is omitted from $L_d(s)$ to retain a fourth order representation for the system. The expression for $L_d(s)$ is from Eq. 1.14

$$L_d(s) = \frac{kh(s^2+S_o s+P_o)(s+z)}{s(s^2+S_p s+P_p)(s+P_z)} \triangleq \frac{khn_d(s)}{d_d(s)} \tag{4.3}$$

The expression for $T_d(s)$ is from Eq. 1.15

$$T_d(s) = \frac{P_r p_{f_1} \; p_{c_z}/z(s+z)}{(s^2+S_r s+P_r)(s+p_{c_z})(s+p_{f_1})} \triangleq \frac{P_r p_{f_1} p_{c_z}/z(s+z)}{D_d(s)} \tag{4.4}$$

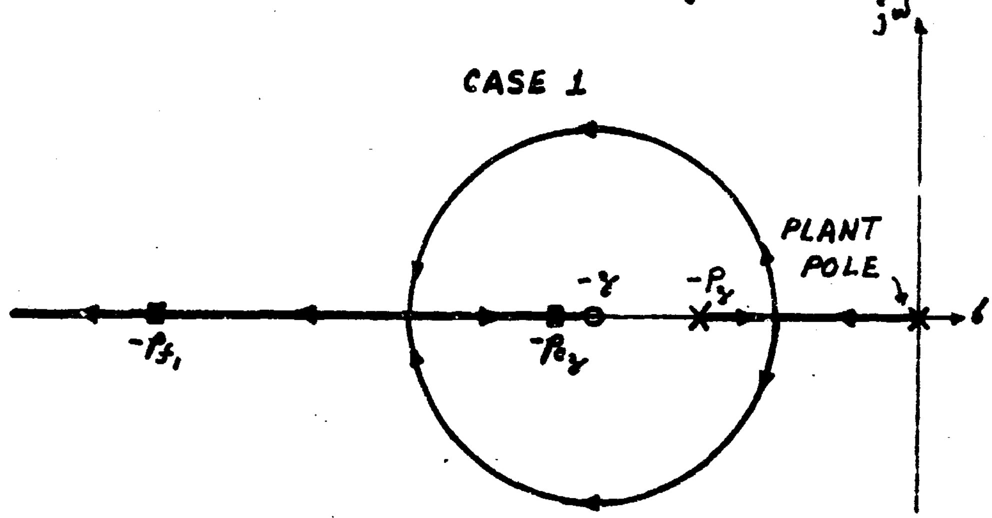
The real axis root locus for this system for the extreme positions of the drifting zero is shown in Fig. 4.4.

The mapping equations for this type of system are derived in the same manner as in Chapter III. The

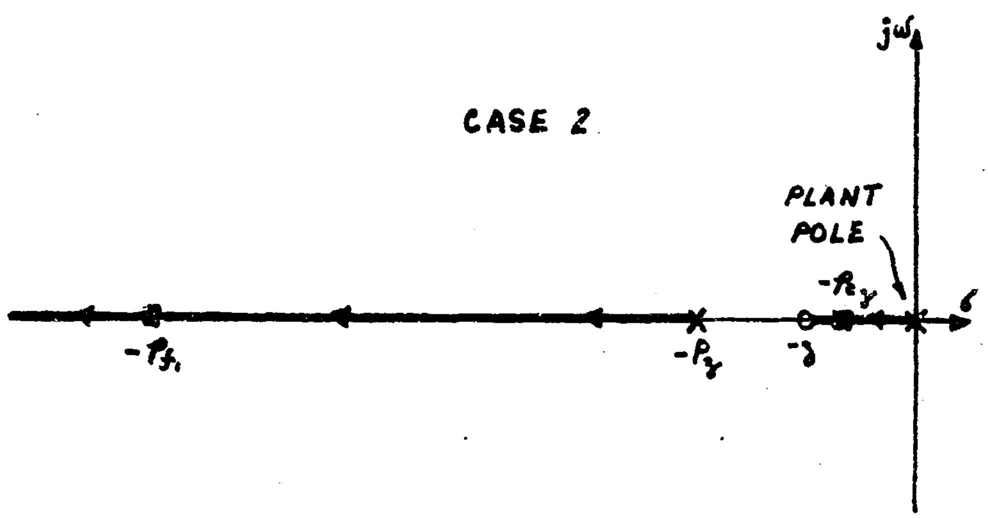
FIG. 4.4 REAL AXIS ROOT LOCUS FOR A FOURTH ORDER SYSTEM WITH A DRIFTING ZERO ON THE REAL AXIS

$$L_d(s) = \frac{kK(s^2+S_o s+P_o)(s+\gamma)}{s(s^2+S_p s+P_p)(s+P_\gamma)}, \quad kK>0$$

CASE 1



CASE 2



characteristic equation of the system is, from Eq. 2.3

$$D_d(s) = d_d(s) + kKn_d(s)$$

$$(s^2+S_r s+P_r)(s+p_{c_z})(s+p_{f_1}) = s(s^2+S_p s+P_p)(s+P_z)+$$
$$kK(s^2+S_o s+P_o)(s+z)$$

or

$$s^4+(S_r+p_{f_1}+p_{c_z})s^3+(p_{f_1}p_{c_z}+S_r(p_{f_1}+p_{c_z})+P_r)s^2+$$
$$(P_r(p_{f_1}+p_{c_z})+S_r p_{f_1}p_{c_z})s+P_r p_{f_1}p_{c_z} =$$
$$s^4+(P_z+S_p+kK)s^3+(P_zS_p+P_p+kK(S_o+z))s^2+(P_zP_p+kK(P_o+S_oz))s$$
$$+kKP_oz \qquad (4.5)$$

Equating the coefficients in Eq. 4.5 yields the following set of equations

$$S_r+p_{f_1}+p_{c_z} = P_z+S_p+kK \qquad (4.6)$$

$$p_{f_1}p_{c_z}+S_r(p_{f_1}+p_{c_z})+P_r = P_zS_p+P_p+kK(S_o+z) \qquad (4.7)$$

$$P_r(p_{f_1}+p_{c_z})+S_r p_{f_1}p_{c_z} = P_zP_p+kK(P_o+S_oz) \qquad (4.8)$$

$$P_r p_{f_1}p_{c_z} = kKP_oz \qquad (4.9)$$

For the problem considered in Chapter III, the parameters relating to the non-dominant closed loop poles on the left hand side of these quations were eliminated by substitution. In this case a slightly different approach is used. Define

$$U \triangleq P_z + S_p + kK \qquad (4.10)$$

$$V \triangleq kKP_o z \qquad (4.11)$$

$$X \triangleq P_z S_p + P_p + kK(S_o + z) \qquad (4.12)$$

$$Y \triangleq P_z P_p + kK(P_o + S_o z) \qquad (4.13)$$

The mapping equations for the dominant closed loop pole region are then

$$U = S_r + p_{f_1} + p_{c_z} \qquad (4.14)$$

$$V = P_r p_{f_1} p_{c_z} \qquad (4.15)$$

$$X = p_{f_1} p_{c_z} + S_r(p_{f_1} + p_{c_z}) + P_r \qquad (4.16)$$

$$Y = P_r(p_{f_1} + p_{c_z}) + S_r p_{f_1} p_{c_z} \qquad (4.17)$$

The total variation in X, ΔX, and the total variation in Y, ΔY, due to the variation in plant poles, i.e. parameters S_p and P_p and the variation in the zero z are given by

$$\Delta X = P_z \Delta S_p + \Delta P_p + kh\Delta z \tag{4.18}$$

$$\Delta Y = P_z \Delta P_p + khS_o \Delta z \tag{4.19}$$

Since kh and S_o are not apriori known, the first approximations to ΔX and ΔY are taken to be

$$\Delta X \cong P_z \Delta S_p + \Delta P_p \tag{4.20}$$

$$\Delta Y \cong P_z \Delta P_p \tag{4.21}$$

These equations are now identical to the mapping equations 3.16 and 3.17 in Chapter III.

4.5 Design Procedure

An outline of the design procedure that will be followed in this problem is as follows:

1.) Map the acceptable dominant closed loop pole region into the U,V plane using Eqs. 4.14 and 4.15 for fixed values of p_{f_1} and p_{c_z}.

2.) Map the acceptable dominant closed loop pole region into the X,Y plane using Eqs. 4.16 and 4.17 for the same fixed values of p_{f_1} and p_{c_z}.

3.) Map the plant pole variation into the $\Delta X,\Delta Y$ plane using Eqs. 4.20 and 4.21 for fixed values of P_z.

4.) Compare the two mappings in (2.3) above. If the plant pole variation in the $\Delta X,\Delta Y$

plane does not fit within the interior of the mapping of the dominant closed loop pole region in the X,Y plane, then the dominant closed loop pole regi n will have to be mapped into both the U,V plane and X,Y plane using a larger value of p_{f_1}.

5.) Solve for the values of kh, S_o and P_o using Eqs. 4.10-4.13 where the values of U,V are obtained from the mapping of the dominant closed loop pole region in the U,V plane and the values of X and Y are obtained from the positioning of the mapping of the plant pole variation within the interior of the mapping of the dominant closed loop pole region in the X,Y plane.

6.) Determine the additional variation of the mapping of the plant pole variation in the X,Y plane by using Eqs. 4.18 and 4.19 and the values of kh and S_o obtained in (5). If this additional variation can not be accommodated within the interior of the mapping of the dominant closed loop pole region in the X,Y plane, then the dominant closed loop pole region must be mapped into both the U,V plane and X,Y plane using a larger value of p_{f_1}.

The next five sections of this chapter elaborate on these steps in the design procedure.

4.6 Mapping of the Dominant Closed Loop Pole Region Into the U,V Plane

The design procedure begins by mapping the dominant closed loop pole region into the U,V plane with parameters p_{f_1} and p_{c_z} using Eq. 4.14 and 4.15.

The parameter p_{f_1}, which is the "far-off" closed loop pole on the real axis, plays the same part as the parameter $\gamma = kKP_o$ did in the design procedure of Chapter III, i.e., large p_{f_1} implies large kK. This can be ascertained from Eq. 4.6 since the parameters S_r, p_{c_z}: P_z and S_p do not have a large variation.

The other parameter in this mapping operation is p_{c_z}, the closed loop pole near the drifting zero. As an approximation to the value of p_{c_z} in the mapping operation, it is assigned values that include the minimum and maximum values of the zero z as well as intermediate values. In practice, this approximation can be improved by performing the mapping operations for $p_{c_z} = z_{max}+\delta_1$ and $p_{c_z}=z_{min}+\delta_2$ where δ_1 and δ_2 are the estimated dipole separations when the zero is at its maximum and minimum positions respectively. The mapping of the dominant closed loop pole region into the U,V plane using Eqs. 4.14 and 4.15 is shown in Fig. 4.5.

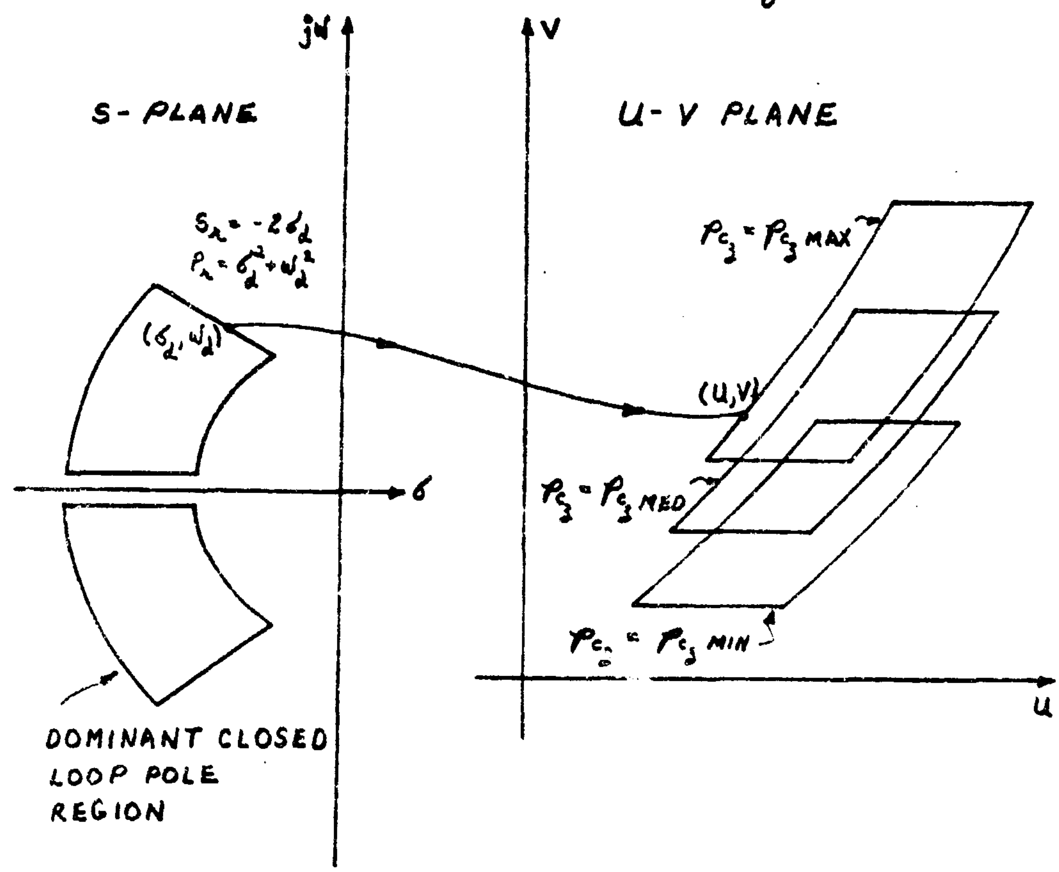
FIG. 4.5 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION IN THE S-PLANE INTO THE U-V PLANE

MAPPING EQUATIONS

$U = S_r + \rho_{f_1} + \rho_{c_\gamma}$

$V = P_r \, \rho_{f_1} \, \rho_{c_\gamma}$

PARAMETERS $\rho_{f_1}, \rho_{c_\gamma}$



From Fig. 4.5 denote the mapping of the dominant closed loop pole region into the U,V plane by Q_i for $p_{c_z} = p_{c_{zi}} \epsilon [z_{min}, z_{max}]$. The mapping of the dominant closed loop pole region into the U,V plane for all values of $p_{c_z} = p_{c_{zi}} \epsilon [z_{min}, z_{max}]$ is shown in Fig. 4.6. The unshaded region shown in Fig. 4.6, denoted by the set Q, can be written as

$$Q = \bigcap_{i=1}^{\infty} Q_i \qquad (4.22)$$

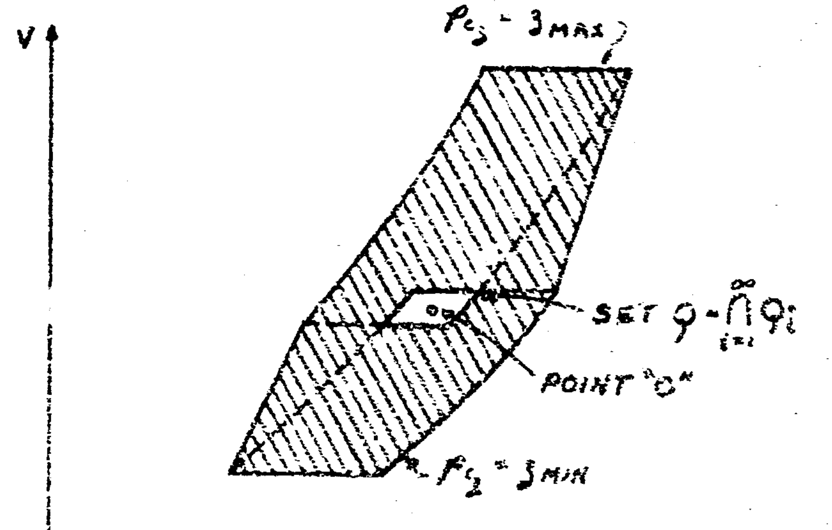
Now any point in Q will map into the acceptable dominant closed loop pole region in the s-plane for all values of $p_{c_z} = p_{c_{zi}} \epsilon [z_{min}, z_{max}]$. This is the same type of argument used in Section 7 of Chapter III. An arbitrary point in Q will be used later to solve for the required values of system gain kh and S_o, P_o which determine the compensation zero position.

In order to insure that the set Q exists, i.e., $Q \neq \Phi$ where Φ is the null set, restrictions must be placed on the magnitude of the zero variation on the real axis and hence p_{c_z}.

Consider the mapping of a point in the dominant closed loop pole region characterized by S_{r_1} and P_{r_1} for $p_{c_z} = p_{c_{zmin}}$. Equations 4.14 and 4.15 map this point into the U,V plane for a fixed value of p_{f_1} as follows:

$$S_{r_1} + p_{f_1} + p_{c_{zmin}} = U_1 \qquad (4.23)$$

FIG. 4.6 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION FOR ALL $\rho_{c_\zeta} \in [\zeta_{MIN}, \zeta_{MAX}]$



$$p_{f_1} p_{c_{zmin}} P_{r_1} = V_1 \qquad (4.24)$$

Now consider another point defined by S_{r_2} and P_{r_2} mapped into the U,V plane for $p_{c_z} = p_{c_{zmax}}$ for the same fixed value of p_{f_1}, i.e.

$$S_{r_2} + p_{f_1} + p_{c_{zmax}} = U_2 \qquad (4.25)$$

$$p_{f_1} p_{c_{zmax}} P_{r_2} = V_2 \qquad (4.26)$$

If the mapping defined by Eqs. 4.23, 4.24 and Eqs. 4.25, 4.26 are to have at least one point in common, the following conditions must be satisfied:

$$U_1 = U_2; \quad V_1 = V_2$$

or $$S_{r_1} - S_{r_2} = p_{c_{zmax}} - p_{c_{zmin}} \qquad (4.27)$$

$$\frac{P_{r_1}}{P_{r_2}} = \frac{p_{c_{zmax}}}{p_{c_{zmin}}} \qquad (4.28)$$

Therefore in order to insure that the set Q exists, there must exist two points contained in the acceptable dominant closed loop pole region defined by S_{r_1}, P_{r_1} and S_{r_2}, P_{r_2} respectively such that Eqs. 4.27 and 4.28 are satisfied. These equations set a limit on the maximum allowable variation in p_{c_z} and hence z in the mapping operation. These restrictions are not overly stringent as will be illustrated in the design example in this chapter.

If the zero variation is large compared to the acceptable dominant closed loop pole region and Eqs. 4.27 and 4.28 are not satisfied, the design procedure presented in this chapter would have to be modified to obtain a design.

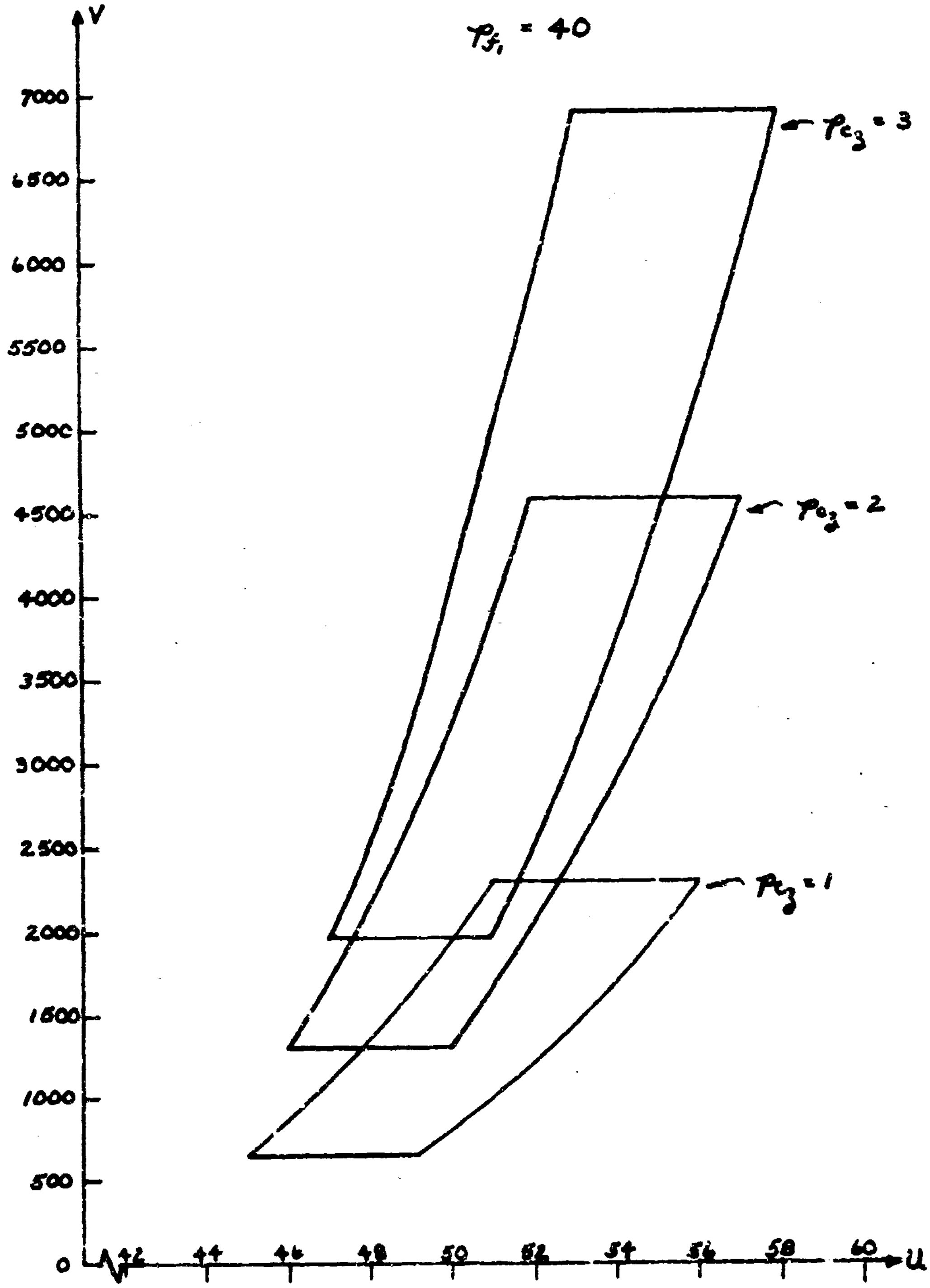
The effect of the parameter p_{c_z} for fixed p_{f_1} on the mapping of the dominant closed loop pole region into the U,V plane is shown in Fig. 4.7. The acceptable dominant closed loop pole region used in this and subsequent mappings in this chapter is the same as that used in the design examples in Chapters II and III and is shown in Fig. 3.26 and in the root tests of Chapter III.

The effect of the parameter p_{f_1} at fixed p_{c_z} on the mapping of the dominant closed loop pole region into the U,V plane is shown in Fig. 4.8.

4.7 Mapping of the Dominant Closed Loop Pole Region Into X,Y Plane

The mapping of the dominant closed loop pole region into the X,Y plane is accomplished using Eqs. 4.16 and 4.17 with the same values of p_{f_1} and p_{c_z} that were used in the mapping in the previous section. This mapping operation is shown in Fig. 4.9. As in Chapter III, the mapping of the plant pole variation will be fitted inside this mapping to solve for the values of kk, S_o and P_o.

FIG. 4.7 THE EFFECT OF THE PARAMETER p_{c_2} ON THE MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION INTO THE U-V PLANE



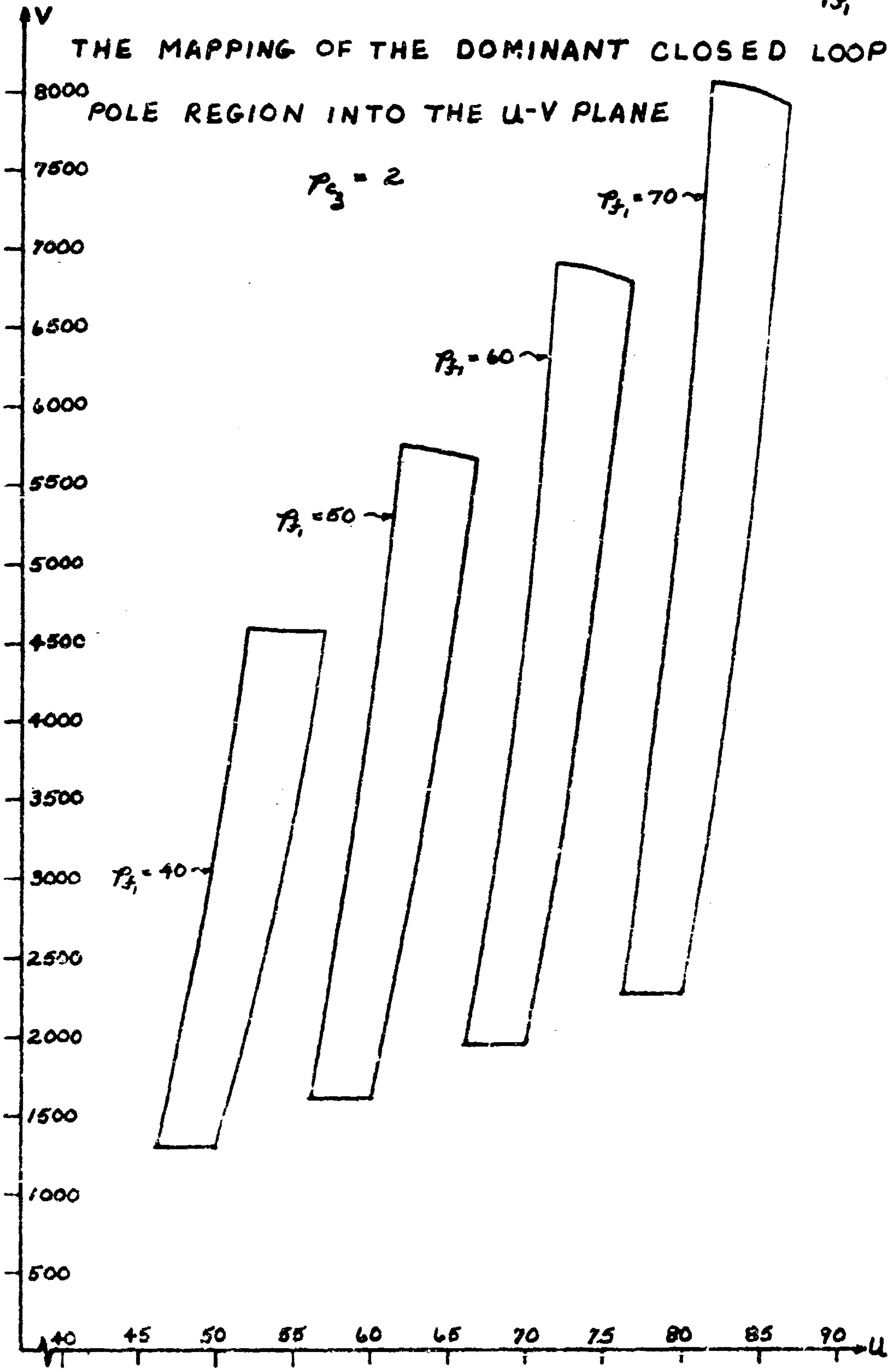


FIG. 4.8 THE EFFECT OF THE PARAMETER P_{f_1} ON THE MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION INTO THE U-V PLANE

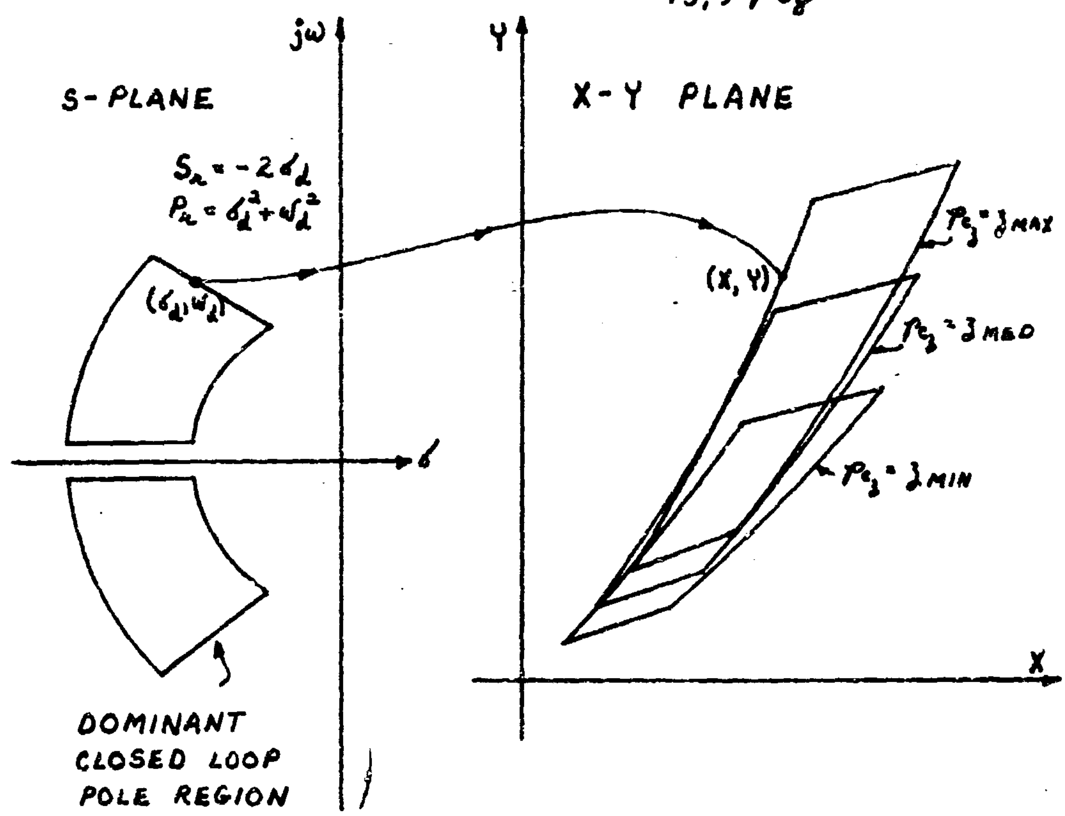
FIG. 4.9 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION IN THE S-PLANE INTO THE X-Y PLANE

MAPPING EQUATIONS

$$X = \rho_{f_1}\rho_{c_\zeta} + S_r(\rho_{f_1} + \rho_{c_\zeta}) + P_r$$

$$Y = P_r(\rho_{f_1} + \rho_{c_\zeta}) + S_r\rho_{f_1}\rho_{c_\zeta}$$

PARAMETERS $\rho_{f_1}, \rho_{c_\zeta}$



The effect of the parameter p_{c_z} for fixed p_{f_1} on the mapping of the dominant closed loop pole region into the X,Y plane is shown in Fig. 4.10. The effect of the parameter p_{f_1} for fixed p_{c_z} on this same mapping is shown in Fig. 4.11.

4.8 Mapping of the Plant Pole Variation Into the $\Delta X,\Delta Y$ Plane

The mapping of the plant pole variation into the $\Delta X,\Delta Y$ plane is accomplished using Eqs. 4.20 and 4.21. Since these equations are identical in form as those in Chapter III, they will not be investigated in detail here. This mapping operation only considers the variation in the plant poles. The variation in the zero on the real axis will be taken into account later in the design. The variation in plant gain may be handled in the same manner as outlined in Chapter III.

The parameter P_z, the open loop pole placed near the drifting zero, occurs in the same manner in Eqs. 4.20 and 4.21 as P_1 did in the mapping of the plant pole variation in Chapter III.

The mapping of the plant pole variation into the $\Delta X,\Delta Y$ plane using Eqs. 4.20 and 4.21 is shown in Fig. 4.12 for various values of P_z. The plant pole variation used in this mapping is identical to that used in Chapter III and is shown in Fig. 3.26. As in Chapter III, the units on this mapping in the $\Delta X,\Delta Y$

FIG. 4.10 THE EFFECT OF THE PARAMETER ρ_{c_3} ON THE MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION INTO THE X-Y PLANE

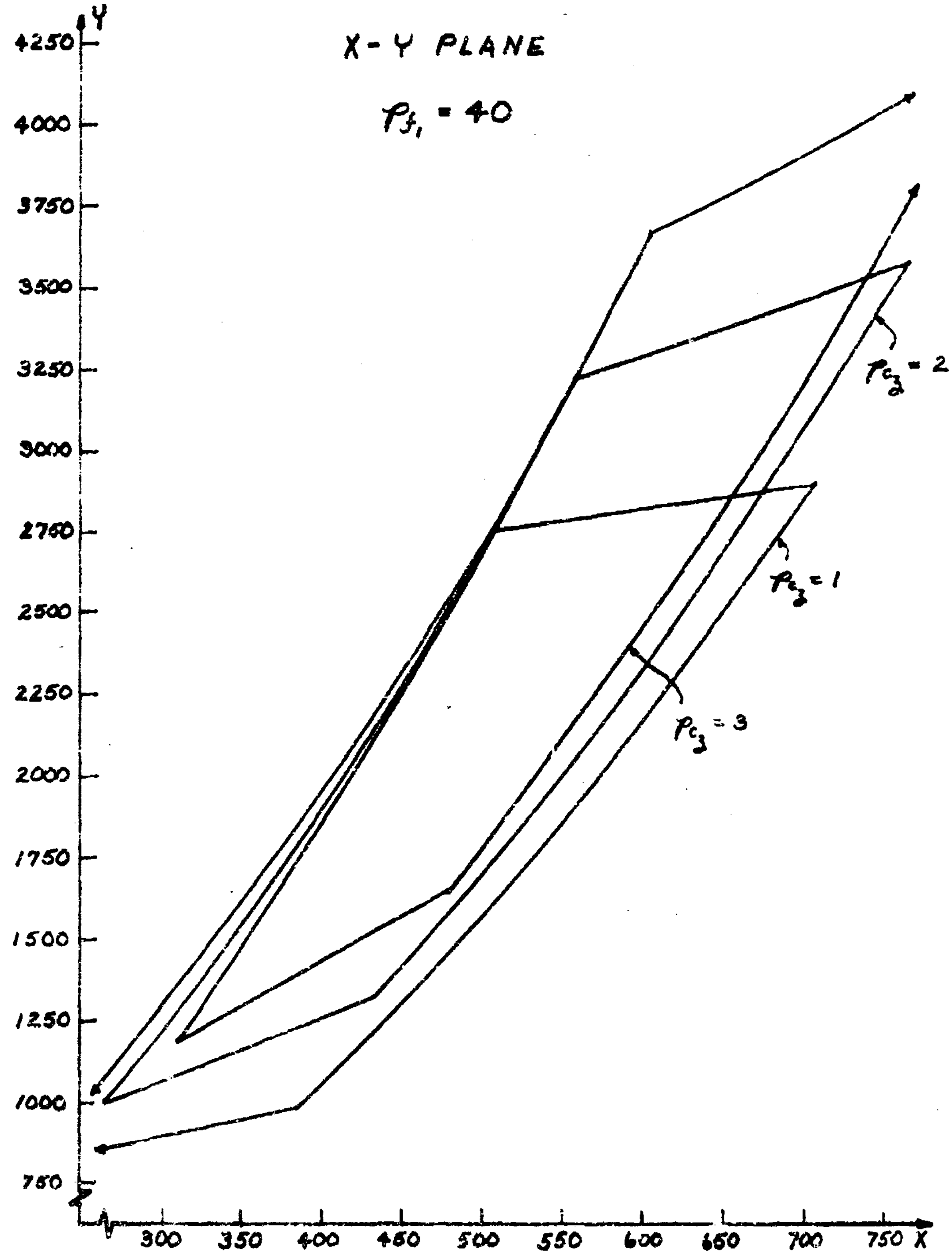


FIG. 4.11 THE EFFECT OF THE PARAMETER ρ_{f_1} ON THE MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION INTO THE X-Y PLANE

$\rho_{c_3} = 2$

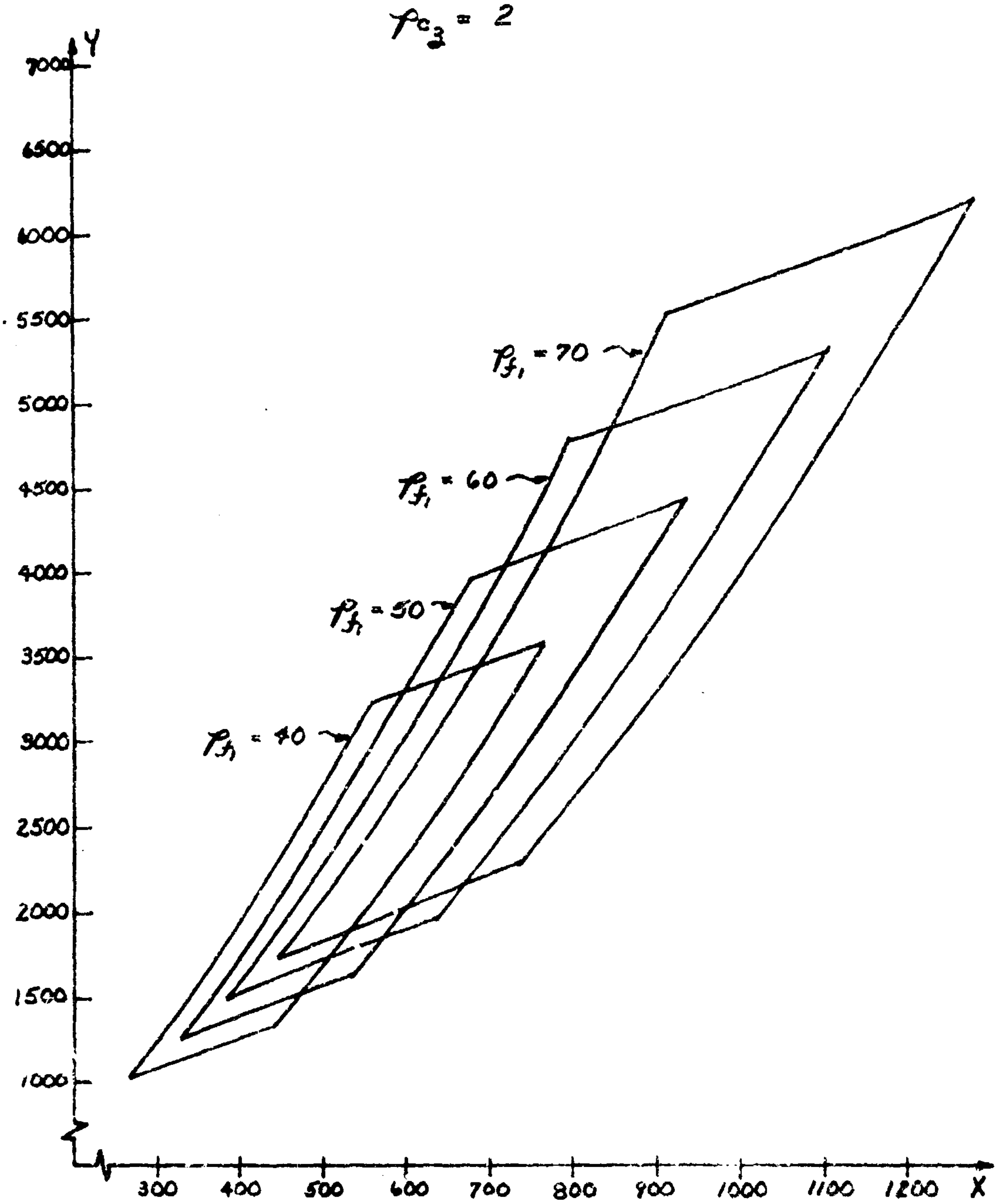
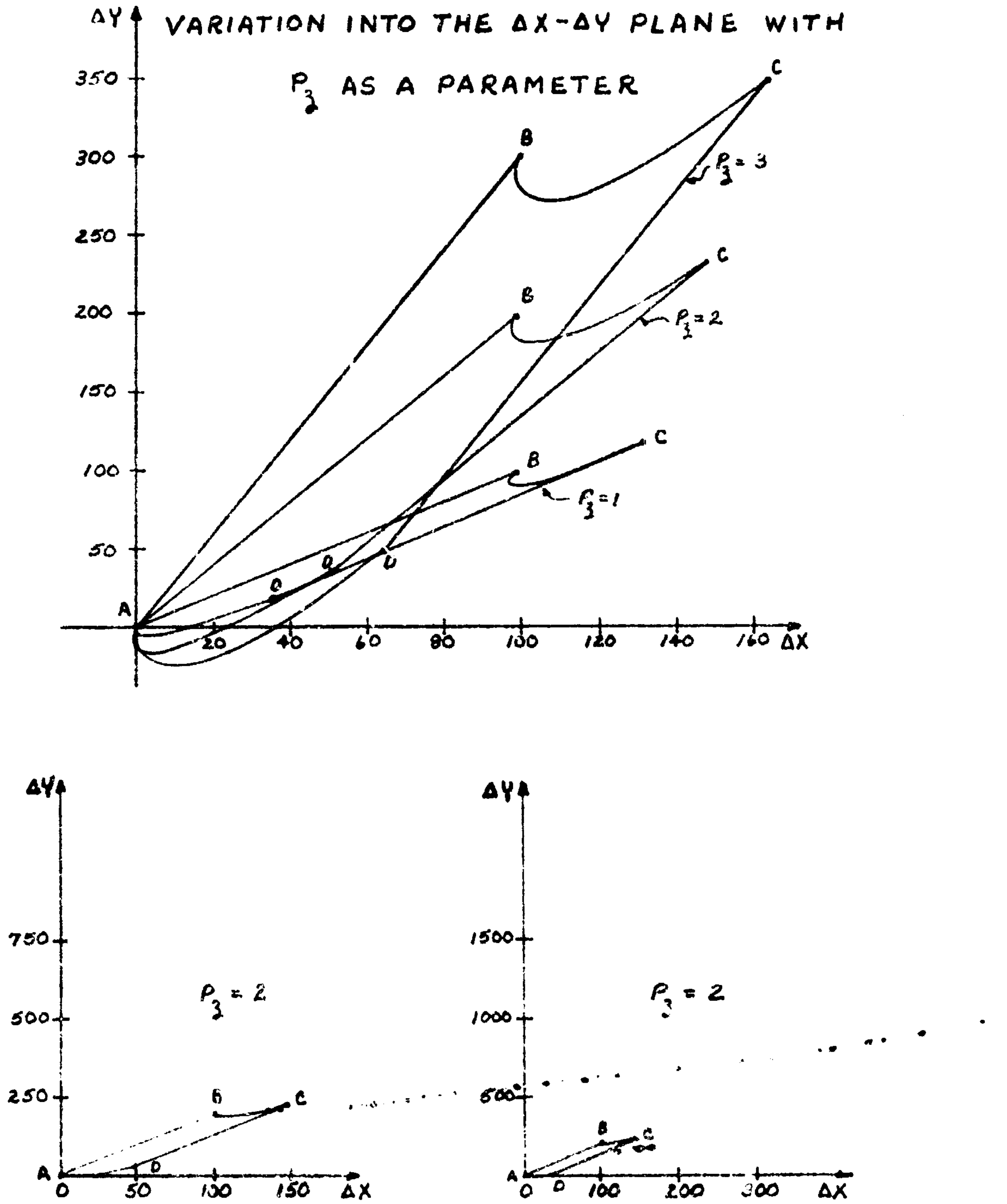


FIG. 4.12 MAPPING OF THE PLANT POLE VARIATION INTO THE ΔX-ΔY PLANE WITH P_3 AS A PARAMETER



plane must be the same as those used in the mapping of the dominant closed loop pole region in the X,Y plane. The mappings of the plant pole variation for $P_z=2$ and comparable units as those used in the mapping of the dominant closed loop pole region into the X,Y plane are also shown in Fig. 4.12.

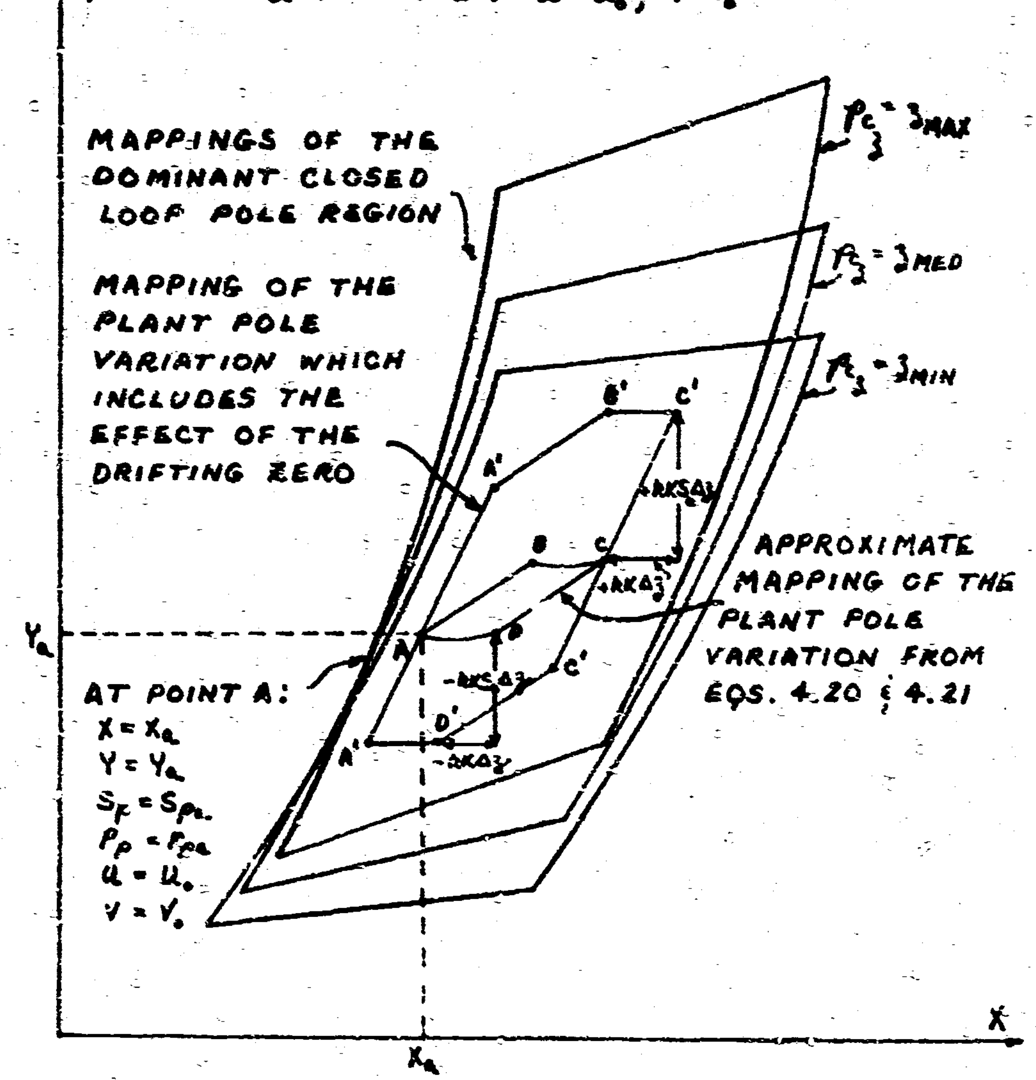
4.9 Calculation of System Gain and Compensation Zero Location

Once the mappings of the dominant closed loop pole region are obtained, an attempt is made to fit the mapping of the plant pole variation in the $\Delta X,\Delta Y$ plane into the interior of the mapping of the dominant closed loop pole region in the X,Y plane as shown in Fig. 4.13. If this is not possible, then the mapping of the dominant closed loop pole region must be performed at higher values of p_{f1}. The mapping of the plant pole variation in the $\Delta X,\Delta Y$ plane is unchanged since P_z is assumed to be fixed at the beginning of the design. Once the mapping of the plant pole variation can be fitted inside the mapping of the dominant closed loop pole region in the X,Y plane, the value of system gain kK and S_o, P_o, the parameters that determine the compensation zero position, can be determined.

The first step in solving for kK, S_o and P_o is to choose an arbitrary point within the set Q of the mapping of the dominant closed loop pole region in the

FIG. 4.13 CALCULATION OF THE SYSTEM GAIN AND THE COMPENSATION ZERO LOCATION FROM THE MAPPINGS OF THE DOMINANT CLOSED LOOP POLE REGION AND THE PLANT POLE VARIATION

FROM POINT "C" IN THE U-V PLANE: $U = U_o$; $V = V_o$



U,V plane. Denote this point by "O" and the values of U and V at this point as U_o and V_o.

At point A in Fig. 4.13, denote the values of S_p, P_p, X and Y as S_{p_a}, P_{p_a}, X_a and Y_a respectively. Since P_z is also known, kK can be found from Eq. 4.10 i.e.,

$$kK = U_o - P_z - S_p \tag{4.29}$$

This leaves Eqs. 4.11, 4.12 and 4.13 to solve for P_o and S_o. The value of z is also an unknown. This value of z is needed to compute the added variation in the mapping of the plant pole variation in the X,Y plane (Eqs. 4.18, 4.19). The effect of the zero variation on the design is considered in the next section.

Using Eq. 4.11 to eliminate z in Eq. 4.12 and 4.13 yields

$$X_a = P_z S_{p_a} + P_{p_a} + kK(S_o + V_o/kKP_o) \tag{4.30}$$

$$Y_a = P_z P_{p_a} + kK(P_o + S_o V_o/kKP_o) \tag{4.31}$$

Solving for P_o in these two equations yields the following third order equation for P_o

$$P_o^3 + \frac{(P_z P_{p_a} - Y_a)P_o^2}{kK} + \frac{V_o(X_a - P_z S_{p_a} - P_{p_a})P_o}{(kK)^2} - \frac{V_o^2}{(kK)^2} = 0 \tag{4.32}$$

Now Eq. 4.32 must have at least one real root which is the one of interest. Note that P_o can not be negative or complex since P_o is related to the magnitude of the compensation zero. Once P_o has been obtained, S_o can be found by solving Eq. 4.30 for S_o, i.e.,

$$S_o = \frac{X_a - P_z S_{p_a} - P_{p_a}}{kK} - \frac{V_o}{kKP_o} \tag{4.33}$$

4.10 Effect of the Zero Variation on the Design

The design is not complete once these values have been obtained since the zero variation Δz in Eqs. 4.18 and 4.19 was neglected in the mapping of the plant pole variation into the $\Delta X, \Delta Y$ plane. This added variation in the mapping of the plant pole variation is taken into account in the X,Y plane by considering that the coordinates of any point on the mapping of the plant pole variation in the X,Y plane could change by as much as

$$\Delta X_z = kK\Delta z \tag{4.34}$$

$$\Delta Y_z = kKS_o\Delta z \tag{4.35}$$

where ΔX_z and ΔY_z is the change in any point on the mapping of the plant pole variation in the X,Y plane due to the variation in the drifting zero only. The zero variation Δz is computed as follows: From Eq. 4.11, the nominal value of z denoted by z_o is

$$z_o = \frac{V_o}{kKP_o} \qquad (4.36)$$

This is the value of z when the mapping of the plant pole variation in the X,Y plane is in its original position. (The position used in the previous section to compute kK, S_o and P_o). The nominal value of z, z_o, will lie somewhere in the interval $[z_{min}, z_{max}]$ because of the values of p_{c_z} used in the mapping operations. The positive change in the coordinates of any point on the mapping of the plant pole variation in the X,Y plane is

$$\Delta X_z^+ = kK(z_{max} - z_o) = +kK\Delta z \qquad (4.37)$$

$$\Delta Y_z^+ = kKS_o(z_{max} - z_o) = +kKS_o\Delta z \qquad (4.38)$$

whereas the negative change is

$$\Delta X_z^- = kK(z_{min} - z_o) = -kK\Delta z \qquad (4.39)$$

$$\Delta Y_z^- = kKS_o(z_{min} - z_o) = -kKS_o\Delta z \qquad (4.40)$$

This variation is shown in Fig. 4.13. Note that ΔX_z^+ is not necessarily equal to ΔX_z^- since z_o may lie anywhere in the interval $[z_{min}, z_{max}]$. This added variation is taken into account by considering the changes in the points A, B, C and D on the boundary of the mapping of the plant pole variation in the X,Y

plane as shown in Fig. 4.13. The maximum possible plant variation is the figure A'A'B'C'C'D' where the primes indicate the new positions of the points A, B, etc. If this added variation can not be fitted within the interior of the mapping of the dominant closed loop pole region in the X,Y plane, then, the dominant closed loop pole region must be mapped into the U,V and X,Y planes using a larger value of p_{f_1}. The mapping of the plant pole variation into the $\Delta X, \Delta Y$ plane remains unchanged as long as P_z is unchanged.

The design should be checked when the total variation can nearly be fitted within the mapping of the dominant closed loop pole region in the X,Y plane since this is an approximate design procedure. The assumption that p_{f_1} remains constant as the dominant closed loop poles vary is only approximately true. The mapping of the dominant closed loop poles at fixed p_{f_1} essentially determines the minimum value that p_{f_1} attains when the actual closed loop poles are obtained by factoring $1+L_d(s) = 0$. The mapping of the dominant closed loop pole region in Chapter III at constant $\gamma = kKP_o$ was completely valid since kK and P_o are fixed if the gain variation is neglected.

4.11 Summary

Unlike the design procedure developed in Chapter III, the design procedure developed here is approximate.

The assumption that p_{f_1} remains fixed as the dominant closed loop poles vary is not strictly valid. For systems with large plant parameter variations, this approximation should lead to an acceptable design since large gain implies a large value of p_{f_1} and thus its effect on the dominant closed loop poles will be slight.

The major difficulty encountered in this problem was determining what effect the zero variation had on the mapping of the plant pole variation (Eqs. 4.18 and 4.19). From these equations, the values of kK and S_o were needed to map the plant pole variation into the $\Delta X, \Delta Y$ plane exactly but this mapping itself was needed to solve for kK and S_o. This difficulty was overcome by first considering the effect of the plant pole variation alone and then checking to determine if this design was adequate to handle the added zero variation. The effect of the zero variation on the mapping of the dominant closed loop pole region was taken into account by the parameter p_{c_z}.

4.12 Design Example

The design example presented here has the same s-domain specifications as the design example in Chapter III (Fig. 3.26) with these two exceptions: The added zero variation along the real axis, i.e. $z_{min} \leq z \leq z_{max}$ where $z_{min}=1$ and $z_{max}=3$. The effect of the "far-off"

pole P_1 is neglected in order to retain a fourth order representation for the system.

The first step in the design is to choose a fixed value of p_{f_1}. In this design procedure, it is difficult to obtain an approximation for the value of p_{f_1} that should be used. Obviously p_{f_1} is going to be relatively far-removed from the acceptable dominant closed loop pole region for large parameter variations. Also the time domain specifications for the problem will probably dictate some minimum value of p_{f_1} (See Fig. 4.1). A value of 40 for p_{f_1} was chosen as the first estimate. Figures 4.14 and 4.15 illustrate the mapping of the dominant closed loop pole region in the U,V and X,Y plane respectively for $p_{f_1} = 40$. In addition, the dipole separations for z_{max}, δ_1, and z_{min}, δ_2, have been estimated at 0.3 and 0.05 respectively (See Section 4.6). Only the mappings for the maximum and minimum values of p_{c_z} are shown in Fig. 4.14 since the set Q is completely defined by these two mappings.

A value of $P_z=2$ will be used since this is midway between the extreme zero positions and is the value used to illustrate the effect of zero variation on the dominant closed loop poles (Figs. 4.2 and 4.3). The mapping of the plant pole variation for $P_z=2$ into the $\Delta X, \Delta Y$ plane is available in Fig. 4.12.

FIG. 4.14 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION INTO THE U-V PLANE

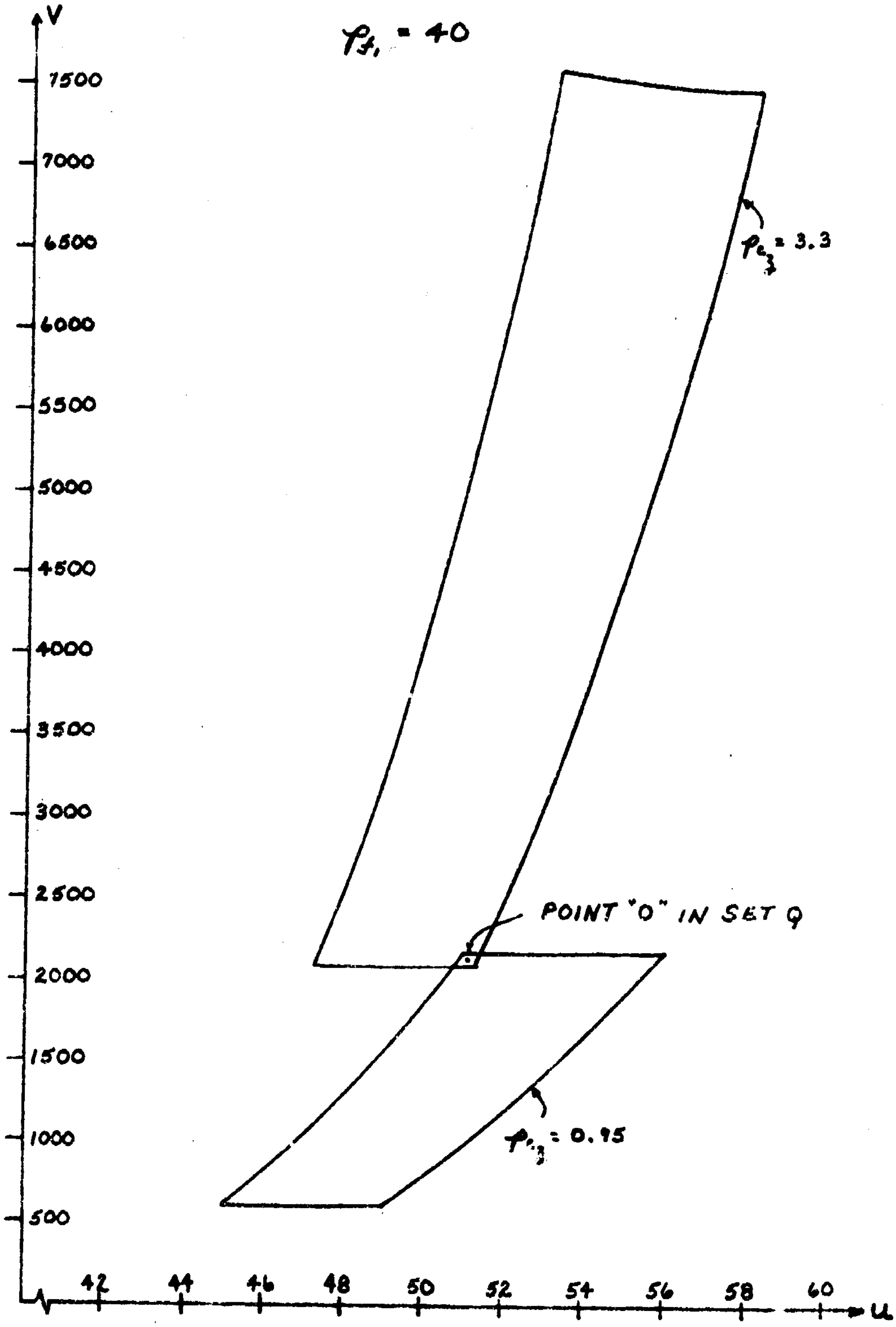


FIG. 4.15 CALCULATION OF THE SYSTEM AND COMPENSATION ZERO POSITION FOR $\varphi_{d_1} = 40$ AND $P_3 = 2$

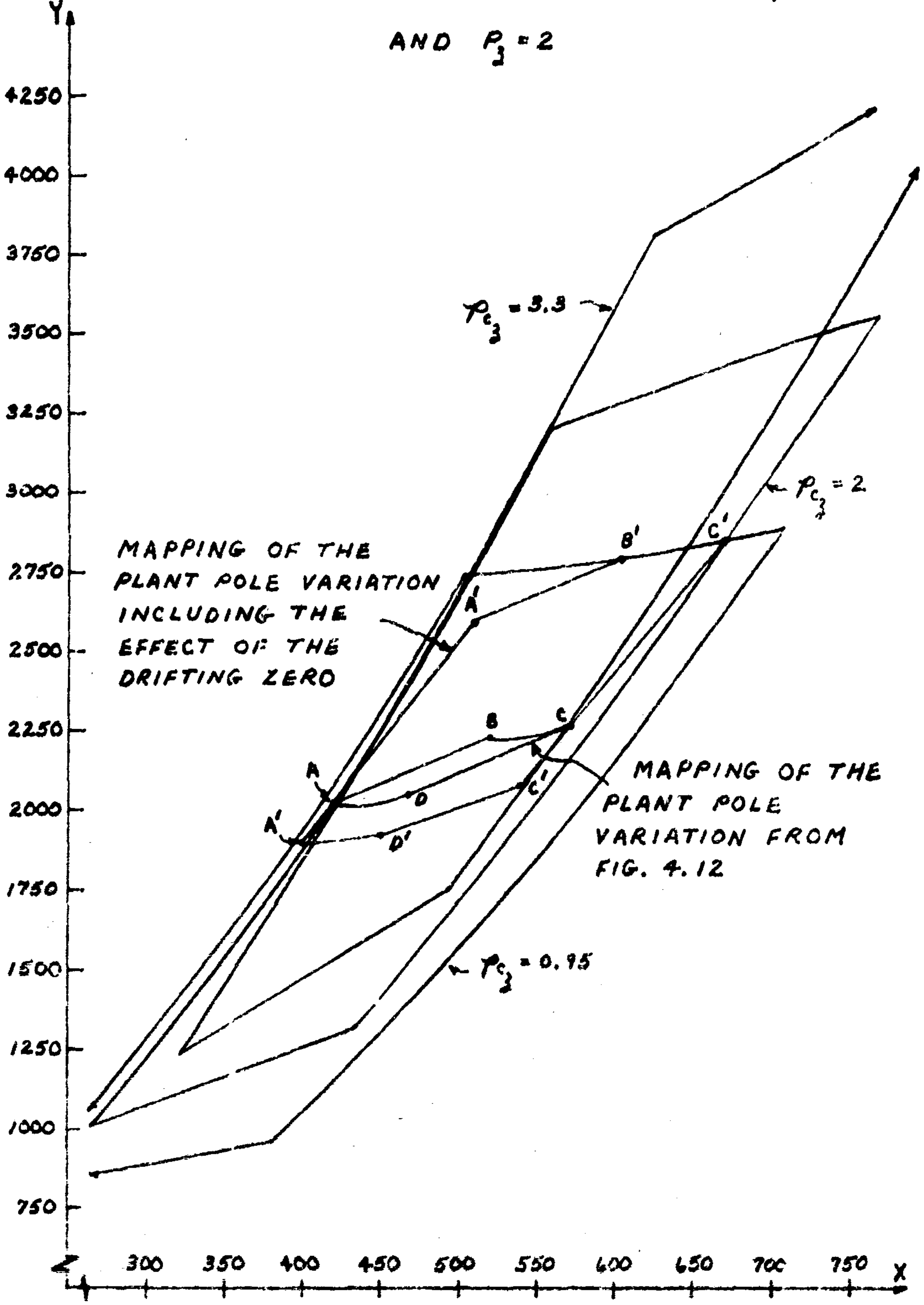


Figure 4.15 shows the mapping of the plant pole variation for $P_z=2$ within the mapping of the dominant closed loop pole region for $p_{f_1}=40$ and $p_{c_z} = 0.95$, 2 and 3.3. At point A in Fig. 4.15

$$X_a = 420 \qquad Y_a = 2025$$

From Fig. 3.26 at point A

$$S_{p_a} = -6 \qquad P_{p_a} = 10$$

From Fig. 4.14, point "0" in the set Q is arbitrarily chosen to have the coordinates

$$U_o = 51 \qquad V_o = 2100$$

Using $P_z=2$, the value of kK can be obtained from Eq. 4.29, i.e.

$$kK = U_o - P_z - S_{p_a} = 51-2+6$$
$$= 55$$

Using Eq. 4.32, the third order equation for P_o is

$$P_o^3 + \frac{(P_z P_{p_a} - Y_a)}{kK} P_o^2 + \frac{V_o(X_a - P_z S_{p_a} - P_{p_a})P_o}{(kK)^2} - \frac{V_o^2}{(kK)^2} = 0$$

$$P_o^3 + \frac{(2(10)-2025)}{55} P_o^2 + \frac{2100(420-2(-6)-10)}{(55)^2} P_o - \frac{(2100)^2}{(55)^2} = 0$$

$$P_o^3 - 36.5P_o^2 + 293P_o - 1460 = 0$$

Factoring the one real positive root of interest using the convergence procedures in Appendix A yields

$$P_o = 27.85$$

S_o is obtained from Eq. 4.33, i.e.

$$S_o = \frac{X_a - P_z S_{Pa} - P_{Pa}}{kK} - \frac{V_o}{kKP_o}$$

$$= \frac{420-2(-6)-10}{55} - \frac{2100}{55(27.85)}$$

$$= 6.31$$

The position of the compensation zeroes is

$$\alpha_z = -\frac{S_o}{2} = -3.105$$

$$\omega_z = \pm\sqrt{P_o^2 - \alpha_z^2} = \pm\sqrt{18.20} = \pm 4.266$$

The design can now be checked to determine if it is adequate for the zero variation. The nominal value of z, z_o, is found from Eq. 4.36, i.e.

$$z_o = \frac{V_o}{kKP_o} = \frac{2100}{55(27.85)} = 1.370$$

The positive variation in the mapping of the plant pole variation in the X, Y plane is found from Eqs. 4.37 and 4.38, i.e.

$$\Delta X_z^+ = kK(z_{max} - z_o) = 55(3.00-1.37)$$

$$= +89.7$$

$$\Delta Y_z^+ = kKS_o(z_{max}-z_o)$$

$$= 55(6.31)(1.63) = +566$$

The negative variation is found from Eqs. 4.39 and 4.40, i.e.

$$\Delta X_z^- = kK(z_{min}-z_o) = 55(1.00-1.37)$$

$$= -20.4$$

$$\Delta Y_z^- = kKS_o(z_{min}-z_o)$$

$$= 55(6.31)(-0.37) = -128.5$$

This added variation is taken into account by considering the change in the coordinates of points A, B, C and D in the X,Y plane as shown in Fig. 4.15. Since this added variation can not be completely accommodated within the mapping of the dominant closed loop pole region, the design is probably not adequate. Using these values of kK, S_o and P_o, the actual closed loop poles were found for the boundary of the plant pole variation shown in Fig. 3.26 by determining the roots of $1+L_d(s)=0$. The results are shown in Fig. 4.16 which confirms that the design is inadequate.

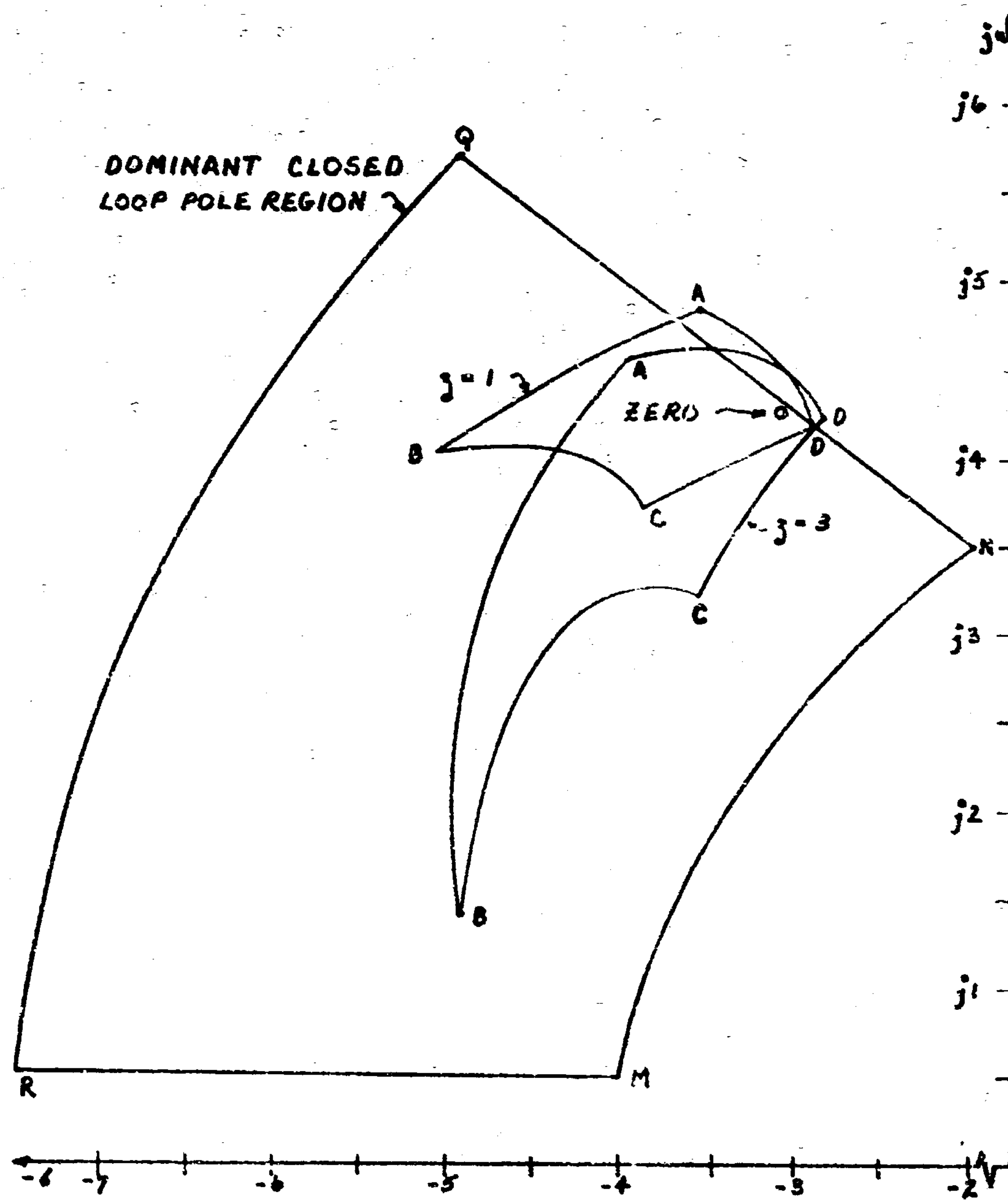
As a check on the design procedure, the position of the compensation zeroes was arbitrarily moved to

$$Z,\overline{Z} = -3.75 \pm j4.25$$

FIG. 4.16 DOMINANT ROOT TEST FOR $\beta_1 = 40$

AND $\beta_3 = 2$

$kK = 55$ $z = -3.105 + j4.266$



in an attempt to force the dominant closed loop poles into their acceptable region. The result using the same value of gain as before is shown in Fig. 4.17 which indicates that this compensation zero position does not yield a satisfactory design either. This implies that the mappings must be performed using a larger value of p_{f_1}.

The next value chosen for p_{f_1} was 50. The mapping of the dominant closed loop pole region into the U,V plane and X,Y plane for this value of p_{f_1} is shown in Figs. 4.18 and 4.19 respectively. The same value of P_z=2 was used so the mapping of the plant pole variation in the $\Delta X,\Delta Y$ plane is unchanged. The mapping of the plant pole variation fitted within the interior of the mapping of the dominant closed loop pole region in the X,Y plane is also shown in Fig. 4.19. At point A in Fig. 4.19

$$X_a = 525 \qquad Y_a = 2500$$

$$S_{p_a} = -6 \qquad P_{p_a} = 10$$

The point "0" in the set Q from Fig. 4.18 is arbitrarily chosen as

$$U_o = 61 \qquad V_o = 2650$$

Using P_z=2, the value of kK from Eq. 4.29 is 65. Using Eq. 4.32, the third order equation for P_o is

FIG. 4.17 DOMINANT ROOT TEST FOR $\varphi_{3_1} = 40$

AND $P_3 = 2$

$kK = 55$ $Z = -3.750 + j\,4.250$

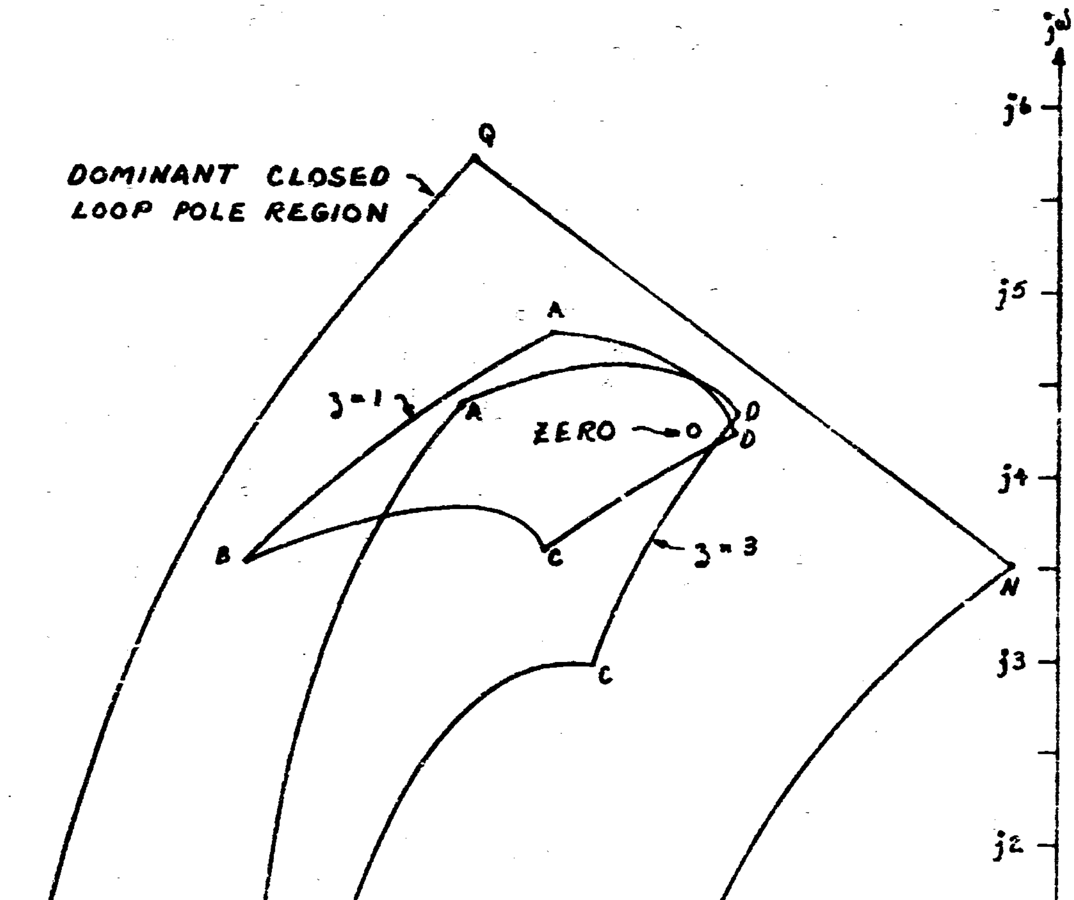


FIG. 4.18 MAPPING OF THE DOMINANT CLOSED LOOP POLE REGION INTO THE U-V PLANE

$\rho_{f_1} = 50$

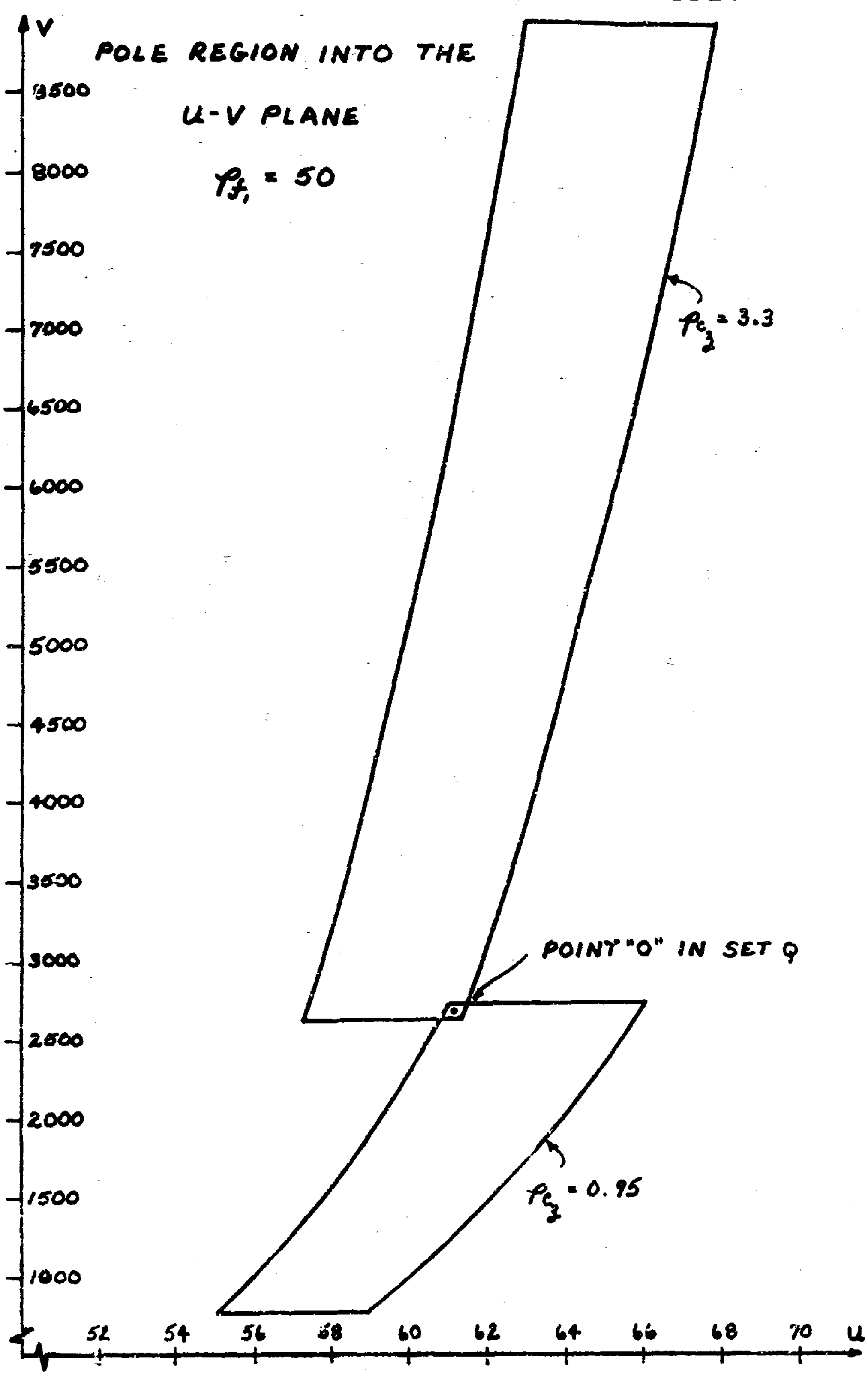
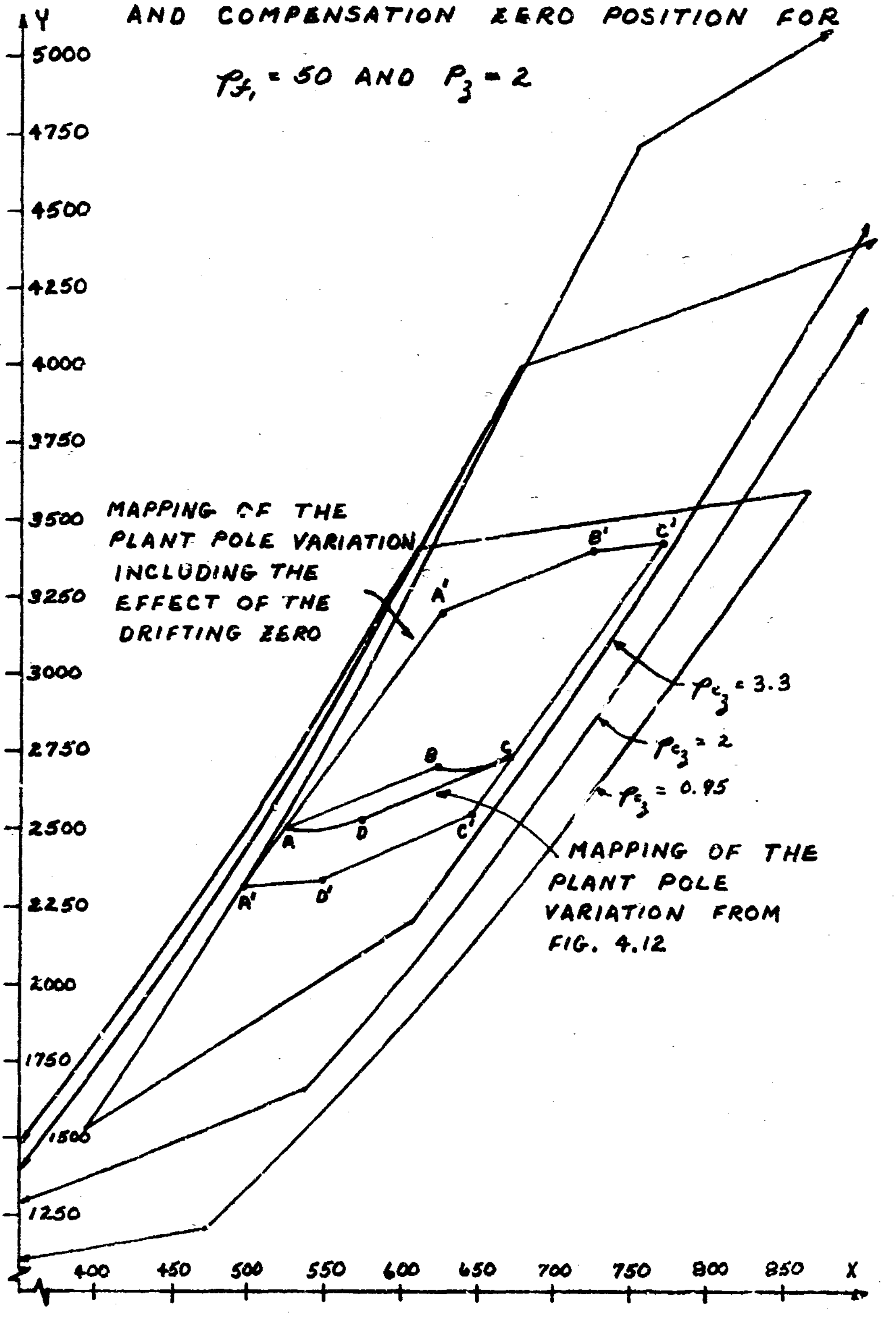


FIG. 4.19 CALCULATION OF THE SYSTEM GAIN AND COMPENSATION ZERO POSITION FOR $\rho_{f_1} = 50$ AND $P_3 = 2$



$$P_o^3 - 38.18\ P_o^2 + 331\ P_o - 1663 = 0$$

The real root of interest is

$$P_o = 28.53$$

S_o from Eq. 4.33 is

$$S_o = 6.68$$

The position of the compensation zeroes is

$$\alpha_z = -3.34$$

$$\omega_z = \pm\ 4.17$$

The nominal value of z, z_o, from Eq. 4.36 is

$$z_o = 1.43$$

The added variation in the mapping of the plant pole variation is found using Eqs. 4.37 - 4.40, i.e.

$$\Delta X_z^+ = +102 \qquad \Delta X_z^- = -28$$

$$\Delta Y_z^+ = +682 \qquad \Delta Y_z^- = -186.5$$

This additional variation is also shown in Fig. 4.19. The additional variation can be easily accommodated within the mapping of the dominant closed loop pole region. The actual closed loop poles for this value of system gain and compensation zero position are shown

in Fig. 4.20. The design is more than adequate to handle the plant pole and zero variation. The difference between the gain of this design and the previous design is approximately 1.7 db.

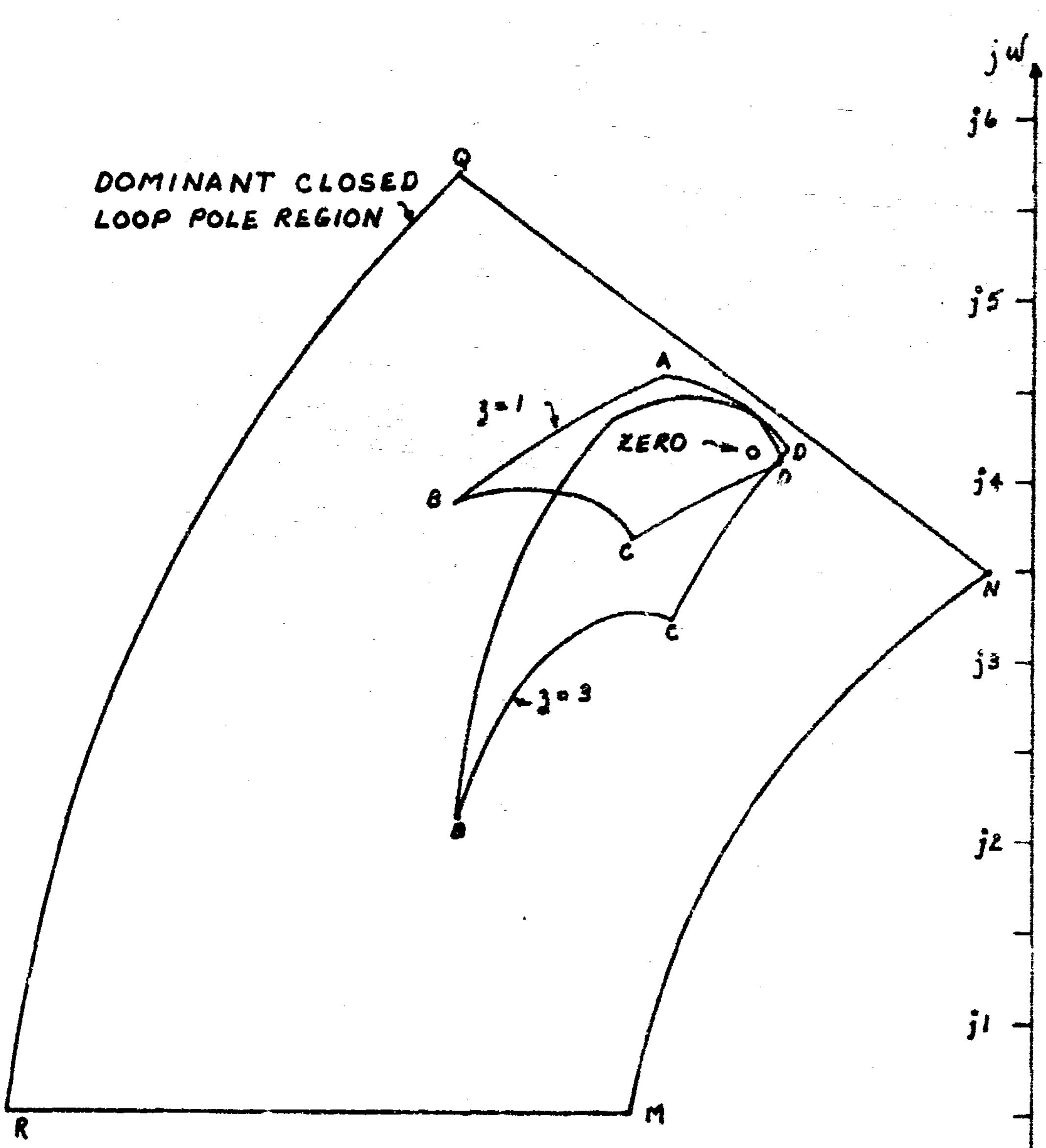
For this particular design, a system gain of less than 65 would probably be adequate. This could be verified by using a value of $p_{f_1} = 45$ for the design procedure. This would result in a system gain of 60. Further reduction in system gain may be possible by varying the value of P_z, the open loop pole used to partially cancel the effect of the drifting zero.

As can be anticipated from the root test shown in Fig. 4.20, the variation in plant gain will not result in the dominant closed loop poles leaving their acceptable region. Using the procedure presented in Chapter III to check the angle of departure of the dominant closed loop poles for the root test shown in Fig. 4.20 confirms this.

FIG. 4.20 DOMINANT ROOT TEST FOR $P_{f_1} = 50$

AND $P_3 = 2$

$kK = 65 \qquad z = -3.34 + j\,4.17$



CHAPTER V

CONCLUSIONS

The design procedures presented in this paper are for fourth order systems with large variations in the plant parameters (gain factor, poles and zeroes). Extension of these design procedures to a fifth order system would be difficult since the mapping equations would be very cumbersome.

Two possibilities for additional work in this area are presented in this chapter which would be a significant improvement over the design procedures developed in this paper.

The first of these is a computor design routine. An initial guess would be made as to the compensation zero location at a fixed value of system gain. By examining the resulting dominant closed loop poles and possibly using a gradient technique, one could determine in what direction the compensation zeroes should be moved to place the dominant closed loop poles within their acceptable region. This is complicated by the fact that the necessary value of system gain is also unknown since for small values of system gain, a design is impossible for any compensation zero location.

Analytic expressions for the boundary of the plant pole variation and the acceptable dominant closed loop pole region would probably be required.

The second possibility in this problem, would be an attempt to obtain an analytic solution for the compensation zero location and necessary value of system gain to place the dominant closed loop poles within a given region in the s-plane for a given region of plant pole variation. Intuitively, one would think that there is a unique value of system gain and compensation zero location such that the closed loop poles lie within their acceptable region and the system gain is minimized. Unfortunately, a solution in closed form appears to be very difficult to obtain in even trivial cases. Analytic expressions for the boundary of the plant pole variation and the acceptable dominant closed loop pole region would certainly be required in this design procedure.

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Appendix A

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APPENDIX A

POLYNOMIAL FACTORING

A.1 Statement of Problem

In many investigations of feedback control systems, polynomials of rather high degree must be factored. In the analysis of feedback control systems, the approximate location of the roots in the complex plane are often known. The problem usually is one of determining if the roots lie within an acceptable region despite variations in gain and/or plant parameters. The polynomial considered in this appendix is of fifth order. A procedure is first developed for extracting one real root from this polynomial. The resulting fourth order polynomial is then factored into the product of two quadratics using Lin's method.[1] This method is applicable whether the fourth order polynomial has real or complex roots. A method is then presented for extracting two real roots from a fourth order polynomial. The nomenclature is chosen to aid in the programming of these procedures on a digital computor. The convergence of any of these methods is not guaranteed.

A.2 Extraction of One Real Root From A Fifth Order Polynomial

The fifth order polynomial is

$$s^5+W1s^4+W2s^3+W3s^2+W4s+W5 \qquad (A.1)$$

The coefficients W1, W2, etc., are considered to be real in all cases.

This procedure is based on the fact that an approximation to the real root is given by the quotient W5/W4. An improvement on this approximation can be made after one long division trial by considering the binomial term in the last subtraction process.[1] With this in mind define

$$V(J) = W5/W4 \qquad J=1 \text{ only} \qquad (A.2)$$

where: $V(J)$ = J^{th} approximation to the real root. The long division operation is shown below.

$$\begin{array}{l|l} & \underline{s^4+A1(J)s^3+A2(J)s^2+A3(J)s+A4(J)} \\ s+V(J) & s^5+W1s^4+W2s^3+W3s^2+W4s+W5 \end{array}$$

$$\underline{s^5+V(J)s^4}$$
$$\{W1-V(J)\}s^4+W2s^3$$

$$\underline{\{W1-V(J)\}s^4+\{W1V(J)-V^2(J)\}s^3}$$
$$\{W2-W1V(J)+V^2(J)\}s^3+W3s^2$$

$$\underline{\{W2-W1V(J)+V^2(J)\}s^3+\{W2V(J)-W1V^2(J)+V^3(J)\}s^2}$$
$$\{W3-W2V(J)+W1V^2(J)-V^3(J)\}s^2+W4s$$

$$\frac{\{W3-W2V(J)+W1V^2(J)-V^3(J)\}s^2+\{W3V(J)-W2V^2(J)+W1V^3(J)-V^4(J)\}s}{\{W4-W3V(J)+W2V^2(J)-W1V^3(J)+V^4(J)\}s+W5}$$

$$\frac{\{W4-W3V(J)+W2V^2(J)-W1V^3(J)+V^4(J)\}s + \{W4V(J)-W3V^2(J)+W2V^3(J)-W1V^4(J)+V^5(J)\}}{\{W5-W4V(J)+W3V^2(J)-W2V^3(J)+W1V^4(J)-V^5(J)\}}$$

The remainder term is the test for convergence. Let

$$X = W5-W4V(J)+W3V^2(J)-W2V^3(J)+W1V^4(J)-V^5(J) \qquad (A.3)$$

If X is not sufficiently small, then the next trial divisor V(J+1) is given by

$$V(J+1) = \frac{W5}{W4-W3V(J)+W2V^2(J)-W1V^3(J)+V^4(J)} \qquad (A.4)$$

The coefficients of the fourth order polynomial are given by

$$A1(J) = W1-V(J) \qquad (A.5)$$

$$A2(J) = W2-W1V(J)+V^2(J) \qquad (A.6)$$

$$A3(J) = W3-W2V(J)+W1V^2(J)-V^3(J) \qquad (A.7)$$

$$A4(J) = W4-W3V(J)+W2V^2(J)-W1V^3(J)+V^4(J) \qquad (A.8)$$

The iteration procedure is continued until the value of X is sufficiently small or until a sufficient number of trials have been made.

A.3 Lin's Method

This method is essentially the same as the previous procedure except the trial divisor is a quadratic term. Let the fourth order polynomial be given by

$$s^4+A1s^3+A2s^2+A3s+A4 \qquad (A.9)$$

Define

$$B1(I)=A3/A2; \quad B2(I)=A4/A2 \quad I=1 \text{ only} \qquad (A.10a, 10b)$$

where: $B1(I)$, $B2(I)$ = coefficients of I^{th} trial quadratic

The trial divisor in the long division operation is given by

$$s^2+B1(I)s+B2(I) \qquad (A.11)$$

Performing the long division operation results in the following

$$s^2+B1(I)s+B2(I)\ \overline{\big)\, s^4+A1s^3+A2s^2+A3s+A4}\ \ \text{quotient: } s^2+C1(I)s+C2(I)$$

$$\underline{s^4+B1(I)s^3+B2(I)s^2}$$

$$\{A1-B1(I)\}s^3+\{A2-B2(I)\}s^2+A3s$$

$$\underline{\{A1-B1(I)\}s^3+\{A1B1(I)-B1^2(I)\}s^2+\{A1B2(I)-B1(I)B2(I)\}s}$$

$$\{A2-B2(I)-A1B1(I)+B1^2(I)\}s^2+\{A3-A1B2(I)+B1(I)B2(I)\}s+A4$$

$$\{A2-B2(I)-A1B1(I)+B1^2(I)\}s^2+\{A2B1(I)-B1(I)B2(I)-A1B1^2(I)+B1^3(I)\}s$$

$$\frac{+\{A2B2(I)-B2^2(I)-A1B1(I)\ B2(I)+B2(I)B1^2(I)\}}{\{A3-A1B2(I)+2B1(I)B2(I)-A2B1(I)+A1B1^2(I) -B1^3(I)\}s+\{A4-A2B2(I)+B2^2(I)+A1B1(I)B2(I) -B2(I)B1^2(I)\}}$$

The two remainder terms are the test for convergence.

Let

$$X=A3-A1B2(I)+2B1(I)B2(I)-A2B1(I)+A1B1^2(I)-B1^3(I) \tag{A.12}$$

$$Y = A4-A2B2(I)+B2^2(I)+A1B1(I)B2(I)-B2(I)B1^2(I) \tag{A.13}$$

If X and Y are not sufficiently small, then the coefficients of the next trial quadratic divisor are given by

$$B1(I+1) = \frac{A3-A1B2(I)+B1(I)B2(I)}{A2-B2(I)-A1B1(I)+B1^2(I)} \tag{A.14}$$

$$B2(I+1) = \frac{A4}{A2-B2(I)-A1B1(I)+B1^2(I)} \tag{A.15}$$

The coefficients of the remaining quadratic are

$$C1(I) = A1-B1(I) \tag{A.16}$$

$$C2(I) = A2-B2(I)-A1B1(I)+B1^2(I) \tag{A.17}$$

The iteration procedure is continued until the values of both X and Y are sufficiently small or a sufficient number of trials have been performed.

A.4 Extraction of Two Real Roots From a Fourth Order Polynomial

This method is applicable when it is known that the fourth order polynomial has at least two real roots.

The fourth order polynomial is given by

$$s^4+A1s^3+A2s^2+A3s+A4$$

As with the fifth order polynomial, define

$$V(J) = A4/A3 \qquad J=1 \text{ only} \tag{A.18}$$

where: $V(J)$ = J^{th} approximation to the real root

The long division operation results in

$$\begin{array}{r|l}
 & \underline{s^3+D1(J)s^2+D2(J)s+D3(J)} \\
s+V(J) & s^4+A1s^3+A2s^2+A3s+A4 \\
 & \underline{s^4+V(J)s^3} \\
 & \{A1-V(J)\}s^3+A2s^2 \\
 & \underline{\{A1-V(J)\}s^3+\{A1V(J)-V^2(J)\}s^2} \\
 & \{A2-A1V(J)+V^2(J)\}s^2+A3s \\
 & \underline{\{A2-A1V(J)+V^2(J)\}s^2+\{A2V(J)-A1V^2(J)+V^3(J)\}s} \\
 & \{A3-A2V(J)+A1V^2(J)-V^3(J)\}s+A4 \\
 & \{A3-A2V(J)+A1V^2(J)-V^3(J)\}s+\{A3V(J)-A2V^2(J) \\
 & \underline{\qquad +A1V^3(J)-V^4(J)\}} \\
 & \{A4-A3V(J)+A2V^2(J)-A1V^3(J)+V^4(J)\}
\end{array}$$

The remainder term is again the test for convergence, i.e., let

$$X = A4-A3V(J)+A2V^2(J)-A1V^3(J)+V^4(J) \tag{A.19}$$

If X is not sufficiently s all, the next trial divisor is

$$V(J+1) = \frac{A4}{A3-A2V(J)+A1V^2(J)-V^3(J)} \tag{A.20}$$

The coefficients of the third order polynomial are

$$D1(J) = A1-V(J) \tag{A.21}$$

$$D2(J) = A2-A1V(J)+V^2(J) \tag{A.22}$$

$$D3(J) = A3-A2V(J)+A1V^2(J)-V^3(J) \tag{A.23}$$

After the first real root is obtained with a sufficient degree of accuracy, the second real root is extracted from the remaining third degree equation. Let the third order polynomial be defined by

$$s^3+C1s^2+C2s+C3 \tag{A.24}$$

Define

$$U(I) = C3/C2 \qquad I=1 \text{ only} \tag{A.25}$$

where: $U(I)$ = I^{th} approximation to real root.

The long division operation results in

$$
\begin{array}{r|l}
 & \underline{s^2+B1\,s+B2} \\
s+U(I) & s^3+C1\,s^2+C2\,s+C3 \\
 & \\
 & \underline{s^3+U(I)s^2} \\
 & \{C1-U(I)\}\,s^2+C2\,s \\
 & \\
 & \underline{\{C1-U(I)\}\,s^2+\{C1U(I)-U^2(I)\}\,s} \\
 & \{C2-C1U(I)+U^2(I)\}\,s+C3 \\
 & \\
 & \underline{\{C2-C1U(I)+U^2(I)\}\,s+\{C2U(I)-C1U^2(I)+U^3(I)\}} \\
 & \{C3-C2U(I)+C1U^2(I)-U^3(I)\}
\end{array}
$$

The test for convergence is

$$X = C3-C2U(I)+C1U^2(I)-U^3(I) \tag{A.26}$$

The next trial divisor, if X is not sufficiently small, is

$$U(I+1) = \frac{C3}{C2-C1U(I)+U^2(I)} \tag{A.27}$$

The coefficients of the quadratic equation are

$$B1(I) = C1-U(I) \tag{A.28}$$

$$B2(I) = C2-C1U(I)+U^2(I) \tag{A.29}$$

In some cases convergence fails in the case of extracting one real root from a third order polynomial. In this case convergence can sometimes be obtained by extracting a quadratic from the third order polynomial.

Define the third order polynomial as

$$s^3+C1s^2+C2s+C3 \qquad (A.30)$$

Also define

$$D1(I) = C2/C1; \quad D2(I) = C3/C1 \quad I=1 \text{ only} \qquad (A.31a, A.31b)$$

where: D1(I), D2(I) = coefficients of the Ith trial quadratic.

The trial divisor in the long division operation is given by

$$s^2+D1(I)s+D2(I) \qquad (A.32)$$

The long division operation results in the following:

$$\begin{array}{r|l}
 & \underline{\quad s+V(I) \quad} \\
s^2+D1(I)s+D2(I) & s^3+C1s^2+C2s+C3 \\
 & \\
 & \underline{s^3+D1(I)s^2+D2(I)s} \\
 & \{C1-D1(I)\}s^2+\{C2-D2(I)\}s+C3 \\
 & \\
 & \{C1-D1(I)\}s^2+\{C1D1(I)-D1^2(I)\}s \\
 & \underline{+\{C1D1(I)-D1(I)D2(I)\}} \\
 & \{C2-D2(I)-C1D1(I)+D1^2(I)\}s \\
 & +\{C3-C1D1(I)+D1(I)D2(I)\}
\end{array}$$

The two remainder terms are a test for convergence. Let

$$X = C2-D2(I)-C1D1(I)+D1^2(I) \qquad (A.33)$$

$$Y = C3-C1D1(I)+D1(I)D2(I) \qquad (A.34)$$

If X and Y are not sufficiently small, then the coefficients of the next trial quadratic are given by

$$D1(I+1) = \frac{C2-D2(I)}{C1-D1(I)} \qquad (A.35)$$

$$D2(I=1) = \frac{C3}{C1-D1(I)} \qquad (A.36)$$

The real root is

$$s = -V(I) = -(C1-D1(I)) \qquad (A.37)$$

A.5 Conclusion

The factoring procedures presented in this appendix may be used for fifth, fourth or third order polynomials possessing either real or complex roots or both. It should be stated again that the convergence of none of these procedures is guaranteed.

APPENDIX B

ANGLE CONTRIBUTION THEOREM

B.1 Statement of Problem

This appendix presents a geometric proof of the statement in Chapter II regarding the angle contribution of two zeroes to a complex pole located in the s-plane. The statement to be proved is as follows: The angle contribution due to two conjugate zeroes to a complex pole is a constant if the zeroes are located on a circular arc drawn through the complex pole and its conjugate and a third point X on the real axis defined by the equation

$$\angle Xp_d^* = \theta_z/2 \qquad (B.1)$$

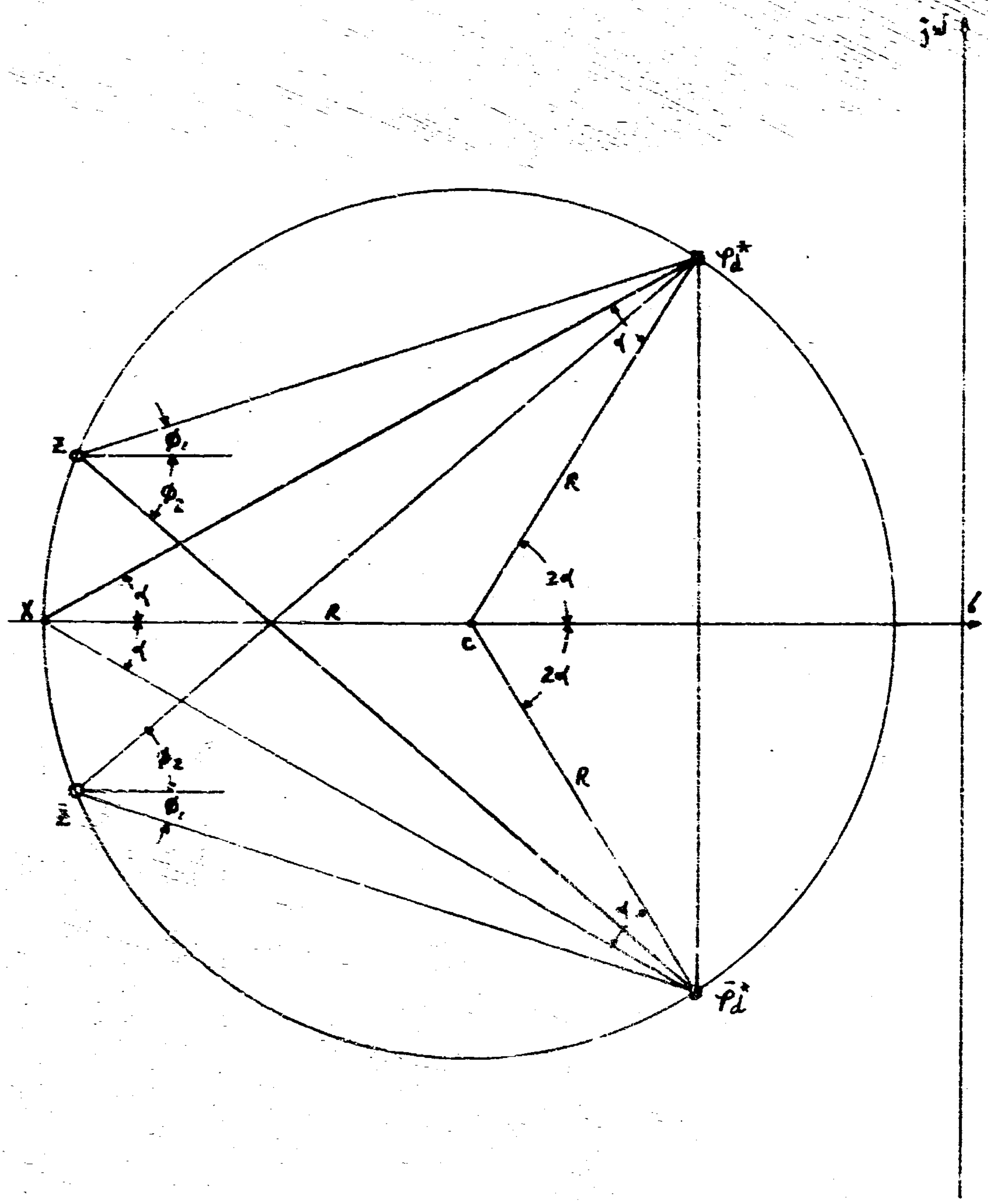
where: p_d^* is the complex pole

θ_z is the angle contribution due to the zeroes

B.2 Geometric Proof

The geometric proof essentially shows that if the compensation zeroes are located on a circular arc through the points $p_d^* X \bar{p}_d^*$, the angle contribution θ_z is a constant independent of the position of the zeroes on the circular arc and that Eq. B.1 is satisfied. The proof is shown in Fig. B.1.

FIG. B-1 GEOMETRIC PROOF OF THE ANGLE OF CONTRIBUTION THEOREM



The proof begins by defining

$$\alpha = \angle Xp_d^* \qquad (B.2)$$

$$\Phi_1 = \angle Zp_d^* \qquad (B.3)$$

$$\Phi_2 = \angle \overline{Z}p_d^* \qquad (B.4)$$

to prove that

$$\Phi_1+\Phi_2 = 2\alpha \qquad (B.5)$$

From Fig. B.1, it is noted that the inscribed angles $p_d^*Z\overline{p}_d^*$, $p_d^*X\overline{p}_d^*$, and $p_d^*\overline{Z}\overline{p}_d^*$ all intercept the same circular arc $p_d^*\overline{p}_d^*$. Therefore from a basic theorem in plane geometry,[1] the following can be stated

$$\angle p_d^*Z\overline{p}_d^* = 1/2\, m\ p_d\overline{p}_d^* \qquad (B.6)$$

$$\angle p_d^*X\overline{p}_d^* = 1/2\, m\ p_d\overline{p}_d^* \qquad (B.7)$$

$$\angle p_d^*\overline{Z}\overline{p}_d^* = 1/2\, m\ p_d\overline{p}_d^* \qquad (B.8)$$

where: $1/2\ m\ p_d\overline{p}_d^*$ = one-half of the measure of the circular arc $p_d^*\overline{p}_d^*$

Also from Fig. B.1,

$$\angle p_d^*Z\overline{p}_d^* = \Phi_1+\Phi_2 \qquad (B.9)$$

$$\angle p_d^*X\overline{p}_d^* = 2\alpha \qquad (B.10)$$

$$\angle p_d^*\overline{Z}\overline{p}_d^* = \Phi_1+\Phi_2 \qquad (B.11)$$

Therefore

$$\Phi_1+\Phi_2 = 2\alpha$$